Frequency of letters in some self-descriptive sequences

Irène Marcovici (LMRS, Univ. de Rouen)

Joint work with **Damien Jamet** (Univ. de Lorraine)
+ Chloé Boisson
+ Shigeki Akiyama (Univ. of Tsukuba), Mai-Linh Trân Công
+ Léo Poirier, Thierry de la Rue (LMRS Rouen)

AofA2026 @ LMU Munich, June 23, 2026

Outline

1. The Oldenburger-Kolakoski sequence
2. Self-descriptive sequences directed by a sequence
   - Random directing sequences
   - Determinstic directing sequences
3. Smooth sequences

The Oldenburger-Kolakoski sequence

Run-length encoding (or *derivation*) operator Δ :

$$u = \underbrace{111}_{3}\underbrace{2}_{1}\underbrace{1}_{1}\underbrace{2222}_{4}\underbrace{111}_{3} \quad \Delta(u) = 31143$$

The Oldenburger-Kolakoski sequence

Run-length encoding (or *derivation*) operator Δ :

$$u = \underbrace{111}_{3}\underbrace{2}_{1}\underbrace{1}_{1}\underbrace{2222}_{4}\underbrace{111}_{3} \quad \Delta(u) = 31143$$

Oldenburger-Kolakoski sequence

It is the unique fixed point of Δ, on the alphabet $\{1, 2\}$, starting with 1.

The Oldenburger-Kolakoski sequence

Run-length encoding (or *derivation*) operator Δ :

$$u = \underbrace{111}_{3}\underbrace{2}_{1}\underbrace{1}_{1}\underbrace{2222}_{4}\underbrace{111}_{3} \quad \Delta(u) = 31143$$

Oldenburger-Kolakoski sequence

It is the unique fixed point of Δ, on the alphabet $\{1, 2\}$, starting with 1.

First terms:

$$\begin{array}{l} \mathcal{O} = \underbrace{1}_{1}\underbrace{22}_{2}\underbrace{11}_{2}\underbrace{2}_{1}\underbrace{1}_{1}\underbrace{22}_{2}\underbrace{1}_{1}\underbrace{22}_{2}\underbrace{11}_{2}\underbrace{2}_{1}\underbrace{11}_{2}\underbrace{22}_{2}\underbrace{1}_{1}\underbrace{2}_{1}\underbrace{11}_{2}\underbrace{2}_{1}\underbrace{1}_{1}\underbrace{22}_{2}\underbrace{11}_{2}\cdots \\ \Delta(\mathcal{O}) = \end{array}$$

The Oldenburger-Kolakoski sequence

Run-length encoding (or *derivation*) operator Δ :

$$u = \underbrace{111}_{3}\underbrace{2}_{1}\underbrace{1}_{1}\underbrace{2222}_{4}\underbrace{111}_{3} \quad \Delta(u) = 31143$$

Oldenburger-Kolakoski sequence

It is the unique fixed point of Δ, on the alphabet $\{1, 2\}$, starting with 1.

First terms:

$$\begin{array}{l} \mathcal{O} = \underbrace{1}_{1}\underbrace{22}_{2}\underbrace{11}_{2}\underbrace{2}_{1}\underbrace{1}_{1}\underbrace{22}_{2}\underbrace{1}_{1}\underbrace{22}_{2}\underbrace{11}_{2}\underbrace{2}_{1}\underbrace{11}_{2}\underbrace{22}_{2}\underbrace{1}_{1}\underbrace{2}_{1}\underbrace{11}_{2}\underbrace{2}_{1}\underbrace{1}_{1}\underbrace{22}_{2}\underbrace{11}_{2}\cdots \\ \Delta(\mathcal{O}) = 1\,2\,2\,1\,1\,2\,1\,2\,2\,1\,2\,2\,1\,1\,2\,1\,1\,2\,2\cdots \end{array}$$

R. Oldenburger, *Exponent Trajectory in Symbolic Dynamics*, Trans Amer. Math. Soc. 46 (1939), 453-466.

W. Kolakoski, *Problem 5304: Self Generating Runs*, Amer. Math. Monthly 72 (1965), 674.

Construction of the sequence

5304. *Proposed by William Kolakoski, Carnegie Institute of Technology*

Describe a simple rule for constructing the sequence

$$1\ 2\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 1\ 2\ 1\ 1\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 1\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 1 \cdots .$$

What is the nth term? Is the sequence periodic?

Recreational publication of W. Kolakoski (1965)

Construction

$\Delta(\mathcal{O}) = \mathcal{O}, \quad \mathcal{O} =$ 1 or 11 ?

Construction of the sequence

5304. *Proposed by William Kolakoski, Carnegie Institute of Technology*

Describe a simple rule for constructing the sequence

1 2 2 1 1 2 1 2 2 1 2 2 1 1 2 1 1 2 2 1 2 1 1 2 1 2 2 1 1 2 1 1 2 1 2 2 1 2 2 1 1 $\cdots$.

What is the nth term? Is the sequence periodic?

Recreational publication of W. Kolakoski (1965)

Construction

$\Delta(\mathcal{O}) = \mathcal{O}, \quad \mathcal{O} = \underbrace{1}_{1}$ ou $\underbrace{11}_{2}$

Construction of the sequence

5304. *Proposed by William Kolakoski, Carnegie Institute of Technology*

Describe a simple rule for constructing the sequence

1 2 2 1 1 2 1 2 2 1 2 2 1 1 2 1 1 2 2 1 2 1 1 2 1 2 2 1 1 2 1 1 2 1 2 2 1 2 2 1 1 $\cdots$.

What is the nth term? Is the sequence periodic?

Recreational publication of W. Kolakoski (1965)

Construction

$\Delta(\mathcal{O}) = \mathcal{O}, \quad \mathcal{O} =$ 1 2 or 1 22 ?

5304. *Proposed by William Kolakoski, Carnegie Institute of Technology*

Describe a simple rule for constructing the sequence

1 2 2 1 1 2 1 2 2 1 2 2 1 1 2 1 1 2 2 1 2 1 1 2 1 2 2 1 1 2 1 1 2 1 2 2 1 2 2 1 1 · · · ·.

What is the nth term? Is the sequence periodic?

Recreational publication of W. Kolakoski (1965)

Construction

$\Delta(\mathcal{O}) = \mathcal{O}, \quad \mathcal{O} = \quad \underbrace{1}_{1}\,\underbrace{\color{red}{2}}_{\color{red}{1}}$ or $\underbrace{1}_{1}\,\underbrace{\color{red}{2}2}_{\color{red}{2}}$

Construction of the sequence

5304. *Proposed by William Kolakoski, Carnegie Institute of Technology*

Describe a simple rule for constructing the sequence

1 2 2 1 1 2 1 2 2 1 2 2 1 1 2 1 1 2 2 1 2 1 1 2 1 2 2 1 1 2 1 1 2 1 2 2 1 2 2 1 1 $\cdots$.

What is the *n*th term? Is the sequence periodic?

Recreational publication of W. Kolakoski (1965)

Construction

$$\Delta(\mathcal{O}) = \mathcal{O}, \qquad \underset{\Delta(\mathcal{O})=}{\mathcal{O} =} \qquad \underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{11}_{2}$$

Construction of the sequence

5304. *Proposed by William Kolakoski, Carnegie Institute of Technology*

Describe a simple rule for constructing the sequence

1 2 2 1 1 2 1 2 2 1 2 2 1 1 2 1 1 2 2 1 2 1 1 2 1 2 2 1 1 2 1 1 2 1 2 2 1 2 2 1 1 $\cdots$.

What is the nth term? Is the sequence periodic?

Recreational publication of W. Kolakoski (1965)

Construction

$$\Delta(\mathcal{O}) = \mathcal{O}, \quad \mathcal{O} = \underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{11}_{2}\,\underbrace{2}_{1}$$

5304. *Proposed by William Kolakoski, Carnegie Institute of Technology*

Describe a simple rule for constructing the sequence

1 2 2 1 1 2 1 2 2 1 2 2 1 1 2 1 1 2 2 1 2 1 1 2 1 2 2 1 1 2 1 1 2 1 2 2 1 2 2 1 1 · · · ·.

What is the nth term? Is the sequence periodic?

Recreational publication of W. Kolakoski (1965)

Construction

$$\Delta(\mathcal{O}) = \mathcal{O}, \qquad \underset{\Delta(\mathcal{O})=}{\mathcal{O} =} \qquad \underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{11}_{2}\,\underbrace{2}_{1}\,\underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{11}_{2}\,\underbrace{2}_{1}\ \ldots$$

Construction of the sequence

> 5304. *Proposed by William Kolakoski, Carnegie Institute of Technology*
>
> Describe a simple rule for constructing the sequence
>
> 1 2 2 1 1 2 1 2 2 1 2 2 1 1 2 1 1 2 2 1 2 1 1 2 1 2 2 1 1 2 1 1 2 1 2 2 1 2 2 1 1 · · · · .
>
> What is the nth term? Is the sequence periodic?

Recreational publication of W. Kolakoski (1965)

Construction

$$\Delta(\mathcal{O}) = \mathcal{O}, \qquad \underset{\Delta(\mathcal{O})=}{\mathcal{O} =} \quad \underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{11}_{2}\,\underbrace{2}_{1}\,\underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{11}_{2}\,\underbrace{2}_{1}\ \dots$$

Periodicity

The sequence $\mathcal{O}$ is not periodic.

State of the art

- Purely morphic sequence? no [Carpi 1993, Lepistö 1993]
- Morphic? open

State of the art

- Purely morphic sequence? no [Carpi 1993, Lepistö 1993]
- Morphic? open
- (Uniformly) recurrent? largely open

State of the art

- Purely morphic sequence? no [Carpi 1993, Lepistö 1993]
- Morphic? open
- (Uniformly) recurrent? largely open
- Factor complexity: largely open
  - $\rightarrow$ $p(n) = O\left(n^{\rho}\right)$ with $\rho = \frac{\log(8)}{\log(4/3)} \approx 7.2283$ [Dekking 1981]
  - $\rightarrow$ Conjecture: $p(n) \sim c\, n^{\delta}$, with $\delta = \frac{\log(3)}{\log(3/2)} \approx 2.7095$
    It would imply not morphic.

State of the art

- Purely morphic sequence? no [Carpi 1993, Lepistö 1993]
- Morphic? open
- (Uniformly) recurrent? largely open
- Factor complexity: largely open

  $\rightarrow$ $p(n) = O\left(n^{\rho}\right)$ with $\rho = \frac{\log(8)}{\log(4/3)} \approx 7.2283$ [Dekking 1981]

  $\rightarrow$ Conjecture: $p(n) \sim c\, n^{\delta}$, with $\delta = \frac{\log(3)}{\log(3/2)} \approx 2.7095$
  It would imply not morphic.
- Frequency of occurences: largely open

  $\rightarrow$ $0.49992 < \varliminf_{n\to\infty} \frac{|\mathcal{O}_0 \cdots \mathcal{O}_{n-1}|_1}{n} \leq \varlimsup_{n\to\infty} \frac{|\mathcal{O}_0 \cdots \mathcal{O}_{n-1}|_1}{n} < 0.50008$
  [Chvátal 1994, Rao 2012]

  $\rightarrow$ We do not know if the limit exists.

A first extension

To other alphabets: example $\Sigma = \{1, 3\}$

The sequence $\mathcal{O}_{3,1} = 3331113331313331113 33\ldots$ is the unique fixed point of Δ starting with 3.

A first extension

To other alphabets: example $\Sigma = \{1, 3\}$

The sequence $\mathcal{O}_{3,1} = 3311133313133311133\ldots$ is the unique fixed point of Δ starting with 3.

$$\mathcal{O}_{3,1} \quad = \quad \underbrace{33}_{A}\underbrace{31}_{B}\underbrace{11}_{C}\underbrace{33}_{A}\underbrace{31}_{B}\underbrace{31}_{B}\underbrace{33}_{A}\underbrace{31}_{B}\underbrace{11}_{C}\underbrace{33}_{A}\ldots$$

$$\sigma : \begin{cases} A & \longmapsto & ABC \\ B & \longmapsto & AB \\ C & \longmapsto & B \end{cases} \qquad M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \pi : \begin{cases} A & \longmapsto & 33 \\ B & \longmapsto & 31 \\ C & \longmapsto & 11 \end{cases}$$

A first extension

To other alphabets: example $\Sigma = \{1, 3\}$

The sequence $\mathcal{O}_{3,1} = 333111333131333111333\ldots$ is the unique fixed point of Δ starting with 3.

$$\mathcal{O}_{3,1} \quad = \quad \underbrace{33}_{A}\underbrace{31}_{B}\underbrace{11}_{C}\underbrace{33}_{A}\underbrace{31}_{B}\underbrace{31}_{B}\underbrace{33}_{A}\underbrace{31}_{B}\underbrace{11}_{C}\underbrace{33}_{A}\ldots$$

$$\sigma : \begin{cases} A & \longmapsto & ABC \\ B & \longmapsto & AB \\ C & \longmapsto & B \end{cases} \qquad M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \pi : \begin{cases} A & \longmapsto & 33 \\ B & \longmapsto & 31 \\ C & \longmapsto & 11 \end{cases}$$

$\mathcal{O}_{3,1}$ is recurrent and the frequencies f_1 et f_3 exist and are algebraic numbers, with $f_1 \approx 0.3972$ et $f_3 \approx 0.6028$.

A first extension

To other alphabets: example $\Sigma = \{1, 3\}$

The sequence $\mathcal{O}_{3,1} = 33311133313133311133 \ldots$ is the unique fixed point of Δ starting with 3.

$$\mathcal{O}_{3,1} \quad = \quad \underbrace{33}_{A}\underbrace{31}_{B}\underbrace{11}_{C}\underbrace{33}_{A}\underbrace{31}_{B}\underbrace{31}_{B}\underbrace{33}_{A}\underbrace{31}_{B}\underbrace{11}_{C}\underbrace{33}_{A}\ldots$$

$$\sigma : \begin{cases} A & \longmapsto & ABC \\ B & \longmapsto & AB \\ C & \longmapsto & B \end{cases} \qquad M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \pi : \begin{cases} A & \longmapsto & 33 \\ B & \longmapsto & 31 \\ C & \longmapsto & 11 \end{cases}$$

$\mathcal{O}_{3,1}$ is recurrent and the frequencies f_1 et f_3 exist and are algebraic numbers, with $f_1 \approx 0.3972$ et $f_3 \approx 0.6028$.

Remark: extension to $\Sigma = \{a, b\}$ with $a + b$ even [Sing 2003 & 2010]

Outline

1. The Oldenburger-Kolakoski sequence
2. Self-descriptive sequences directed by a sequence
   - Random directing sequences
   - Determinstic directing sequences
3. Smooth sequences

Notion of directing sequence

A look back at the construction of the Oldenburger-Kolakoski sequence:

$T = 1$
$\mathcal{O}_{1,2} = \underbrace{1}_{1}$

Blocks are filled alternatively with 1's and 2's.

The sequence $T = (12)^\infty$ is the **directing sequence** of $\mathcal{O}_{1,2}$.

- The n^{th} block of $\mathcal{O}_{1,2}$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_{1,2}$ gives the length of the n^{th} block, that is, the number of times we concatenate the letter T_n.

Notion of directing sequence

A look back at the construction of the Oldenburger-Kolakoski sequence:

$$T = 1, 2$$
$$\mathcal{O}_{1,2} = \underbrace{1}_{1}\,\underbrace{22}_{2}$$

Blocks are filled alternatively with 1's and 2's.

The sequence $T = (12)^\infty$ is the **directing sequence** of $\mathcal{O}_{1,2}$.

- The n^{th} block of $\mathcal{O}_{1,2}$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_{1,2}$ gives the length of the n^{th} block, that is, the number of times we concatenate the letter T_n.

Notion of directing sequence

A look back at the construction of the Oldenburger-Kolakoski sequence:

$$T = 1, 2$$
$$\mathcal{O}_{1,2} = \underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{??}_{2}$$

Blocks are filled alternatively with 1's and 2's.

The sequence $T = (12)^\infty$ is the **directing sequence** of $\mathcal{O}_{1,2}$.

- The n^{th} block of $\mathcal{O}_{1,2}$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_{1,2}$ gives the length of the n^{th} block, that is, the number of times we concatenate the letter T_n.

Notion of directing sequence

A look back at the construction of the Oldenburger-Kolakoski sequence:

$T = 1, 2, {\color{blue}1}$
$\mathcal{O}_{1,2} = \underbrace{1}_{1}\underbrace{22}_{2}\ \underbrace{{\color{blue}11}}_{2}$

Blocks are filled alternatively with 1's and 2's.

The sequence $T = (12)^\infty$ is the **directing sequence** of $\mathcal{O}_{1,2}$.

- The n^{th} block of $\mathcal{O}_{1,2}$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_{1,2}$ gives the length of the n^{th} block, that is, the number of times we concatenate the letter T_n.

Notion of directing sequence

A look back at the construction of the Oldenburger-Kolakoski sequence:

$$T = 1, 2, 1$$
$$\mathcal{O}_{1,2} = \underbrace{1}_{1}\ \underbrace{22}_{2}\ \underbrace{11}_{2}\ \underbrace{?}_{1}\ \underbrace{?}_{1}$$

Blocks are filled alternatively with 1's and 2's.

The sequence $T = (12)^\infty$ is the **directing sequence** of $\mathcal{O}_{1,2}$.

- The n^{th} block of $\mathcal{O}_{1,2}$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_{1,2}$ gives the length of the n^{th} block, that is, the number of times we concatenate the letter T_n.

Notion of directing sequence

A look back at the construction of the Oldenburger-Kolakoski sequence:

$$T = 1, 2, 1, 2$$
$$\mathcal{O}_{1,2} = \underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{11}_{2}\,\underbrace{2}_{1}\,\underbrace{?}_{1}$$

Blocks are filled alternatively with 1's and 2's.

The sequence $T = (12)^\infty$ is the **directing sequence** of $\mathcal{O}_{1,2}$.

- The n^{th} block of $\mathcal{O}_{1,2}$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_{1,2}$ gives the length of the n^{th} block, that is, the number of times we concatenate the letter T_n.

Example with an arbitrary directing sequence:

$T = ?, ?, ?, ?, ? \ldots$
$\mathcal{O}_T =$

- The n^{th} "block" of $\mathcal{O}_T$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_T$ gives the length of the n^{th} "block", that is, the number of times we concatenate the letter T_n.

Example with an arbitrary directing sequence:

$T = 2, ?, ?, ?, ? \ldots$
$\mathcal{O}_T = \underbrace{22}_{2}$

- The n^{th} "block" of $\mathcal{O}_T$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_T$ gives the length of the n^{th} "block", that is, the number of times we concatenate the letter T_n.

Notion of directing sequence

Example with an arbitrary directing sequence:

$T = 2, ?, ?, ?, ? \ldots$
$\mathcal{O}_T = \underbrace{22}_{2}\underbrace{??}_{2}$

- The n^{th} "block" of $\mathcal{O}_T$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_T$ gives the length of the n^{th} "block", that is, the number of times we concatenate the letter T_n.

Notion of directing sequence

Example with an arbitrary directing sequence:

$T = 2, 1, ?, ?, ? \dots$
$\mathcal{O}_T = \underbrace{22}_{2}\underbrace{11}_{2}$

- The n^{th} "block" of $\mathcal{O}_T$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_T$ gives the length of the n^{th} "block", that is, the number of times we concatenate the letter T_n.

Notion of directing sequence

Example with an arbitrary directing sequence:

$T = 2, 1, ?, ?, ? \ldots$

$\mathcal{O}_T = \underbrace{22}_{2}\underbrace{\color{red}{11}}_{2}\underbrace{?}_{\color{red}{1}}\underbrace{?}_{\color{red}{1}}$

- The n^{th} "block" of $\mathcal{O}_T$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_T$ gives the length of the n^{th} "block", that is, the number of times we concatenate the letter T_n.

Example with an arbitrary directing sequence:

$$T = 2, 1, 2, ?, ? \ldots$$
$$\mathcal{O}_T = \underbrace{22}_{2}\underbrace{11}_{2}\underbrace{2}_{1}\underbrace{?}_{1}$$

- The n^{th} "block" of $\mathcal{O}_T$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_T$ gives the length of the n^{th} "block", that is, the number of times we concatenate the letter T_n.

Notion of directing sequence

Example with an arbitrary directing sequence:

$$T = 2, 1, 2, ?, ? \ldots$$
$$\mathcal{O}_T = \underbrace{22}_{2}\underbrace{11}_{2}\underbrace{2}_{1}\underbrace{?}_{1}\underbrace{??}_{2}$$

- The n^{th} "block" of $\mathcal{O}_T$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_T$ gives the length of the n^{th} "block", that is, the number of times we concatenate the letter T_n.

Example with an arbitrary directing sequence:

$T = 2, 1, 2, 2, ? \dots$

$\mathcal{O}_T = \underbrace{22}_{2}\underbrace{11}_{2}\underbrace{2}_{1}\underbrace{2}_{1}\underbrace{??}_{2}$

- The n^{th} "block" of $\mathcal{O}_T$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_T$ gives the length of the n^{th} "block", that is, the number of times we concatenate the letter T_n.

Notion of directing sequence

Example with an arbitrary directing sequence:

$T = 2, 1, 2, 2, ? \ldots$

$\mathcal{O}_T = \underbrace{22}_{2}\,\underbrace{11}_{2}\,\underbrace{2}_{1}\,\underbrace{2}_{1}\,\underbrace{??}_{2}\,\underbrace{??}_{2}$

- The n^{th} "block" of $\mathcal{O}_T$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_T$ gives the length of the n^{th} "block", that is, the number of times we concatenate the letter T_n.

Example with an arbitrary directing sequence:

$T = 2, 1, 2, 2, 1 \ldots$

$\mathcal{O}_T = \underbrace{22}_{2}\,\underbrace{11}_{2}\,\underbrace{2}_{1}\,\underbrace{2}_{1}\,\underbrace{11}_{2}\,\underbrace{??}_{2}$

- The n^{th} "block" of $\mathcal{O}_T$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_T$ gives the length of the n^{th} "block", that is, the number of times we concatenate the letter T_n.

Example with an arbitrary directing sequence:

$T = 2, 1, 2, 2, 1 \ldots$

$\mathcal{O}_T = \underbrace{22}_{2}\underbrace{11}_{2}\underbrace{2}_{1}\underbrace{2}_{1}\underbrace{11}_{2}\underbrace{??}_{2}\underbrace{?}_{1}\underbrace{?}_{1}$

- The n^{th} "block" of $\mathcal{O}_T$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_T$ gives the length of the n^{th} "block", that is, the number of times we concatenate the letter T_n.

Notion of directing sequence

Example with an arbitrary directing sequence:

$$T = 2, 1, 2, 2, 1 \ldots$$
$$\mathcal{O}_T = \underbrace{22}_{2}\underbrace{11}_{2}\underbrace{2}_{1}\underbrace{2}_{1}\underbrace{\color{red}{11}}_{2}\underbrace{??}_{2}\underbrace{?}_{\color{red}{1}}\underbrace{?}_{\color{red}{1}}$$

- The n^{th} "block" of $\mathcal{O}_T$ is filled with the letter T_n.
- The n^{th} letter of $\mathcal{O}_T$ gives the length of the n^{th} "block", that is, the number of times we concatenate the letter T_n.

For any **directing sequence** T, we can define a **resulting sequence** $\mathcal{O}_T$.

Remark: the sequence $\mathcal{O}_T$ is no more differentiable.

Python program

```
def O(T):
  X = []
  k = 0
  for x in T:
    X += [x] # concatenate 'x' at the end of X
    X += [x]*(X[k]-1) # concatenate 'X[k]-1' copie(s) of x
    k += 1
  return X

```

Outline

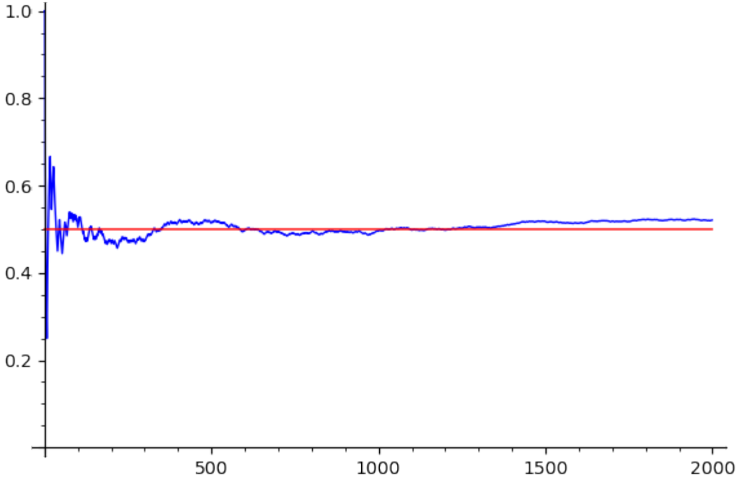
1. The Oldenburger-Kolakoski sequence
2. Self-descriptive sequences directed by a sequence
   - Random directing sequences
   - Determinstic directing sequences
3. Smooth sequences

Independent coin tosses

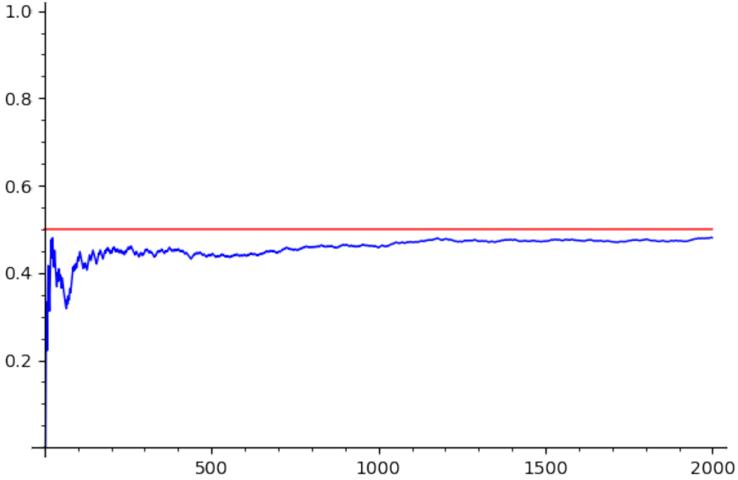
- Directing sequence $\mathbb{T} = (T_n)_{n\in\mathbb{N}^\star}$ made of unif. independent variables:
$$\mathbb{P}(T_n = 1) = \mathbb{P}(T_n = 2) = \frac{1}{2}.$$
- What is the distribution of the sequence $\mathbb{X} = (X_n)_{n\in\mathbb{N}^\star}$ defined by $\mathbb{X} = \mathcal{O}_{\mathbb{T}}$?
  - What is the evolution of the frequency of 1's in $\mathbb{X}$?
  - What is the value of $\mathbb{P}(X_n = 1)$?

Independent coin tosses

Evolution of the frequency of 1's for an example of realisation (1)

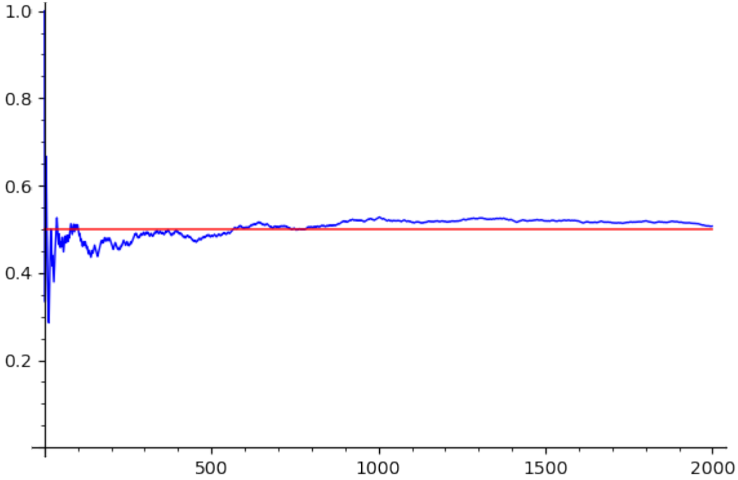


Evolution of the frequency of 1's for an example of realisation (2)



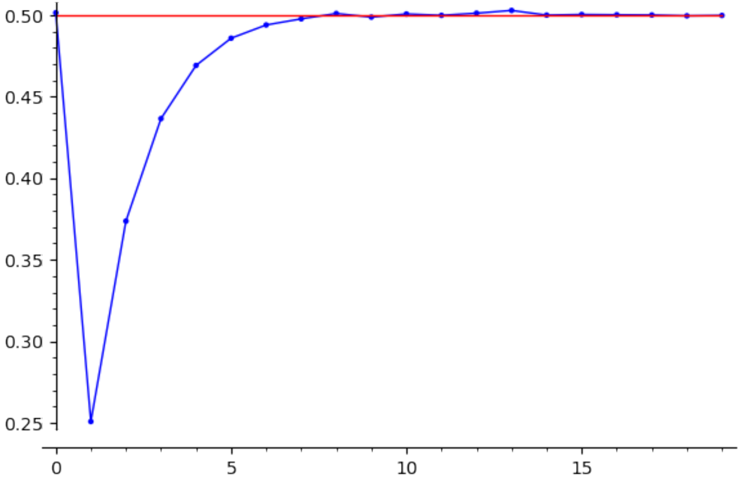
Independent coin tosses

Evolution of the frequency of 1's for an example of realisation (3)

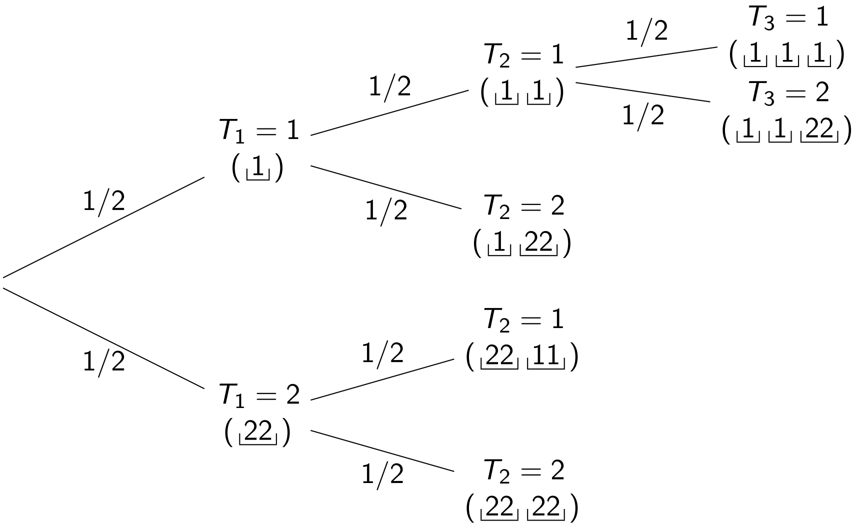


Independent coin tosses

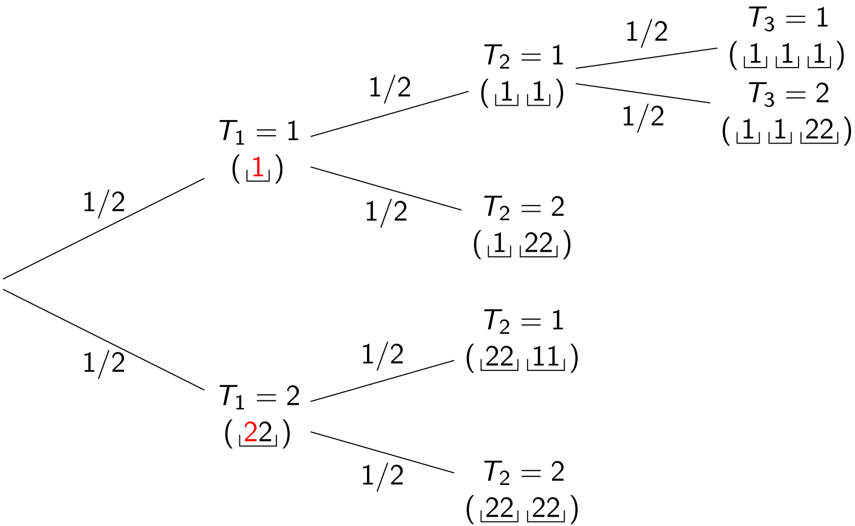
Proportion of 1's in each of the first 20 letters,
for 100 000 realisations



Independent coin tosses

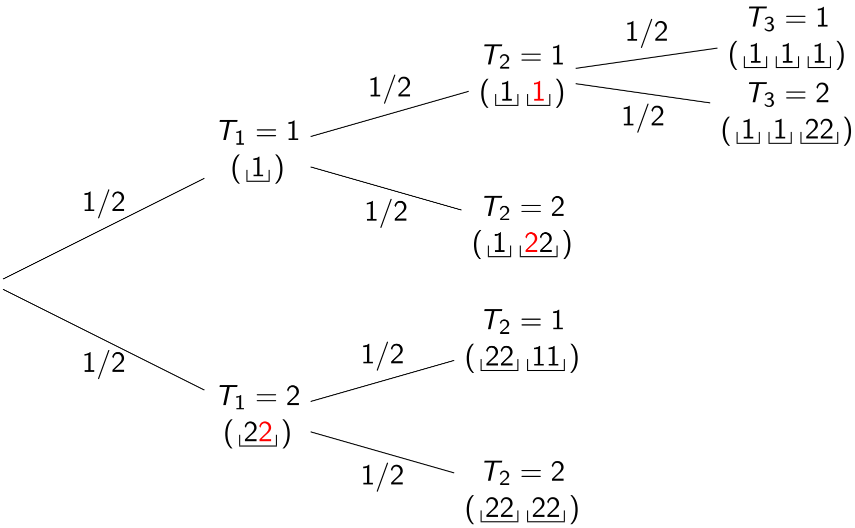


Independent coin tosses



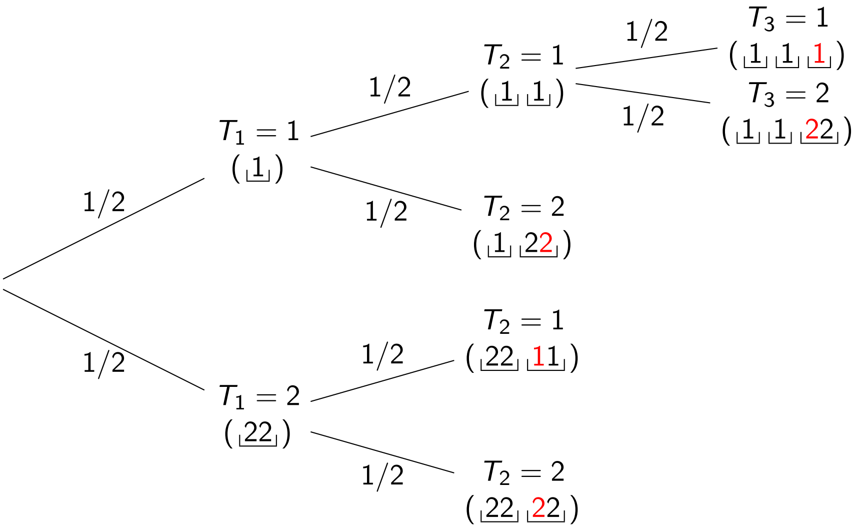
$\mathbb{P}(X_1 = 1) = \frac{1}{2}$

Independent coin tosses



$\mathbb{P}(X_1 = 1) = \frac{1}{2}$ $\mathbb{P}(X_2 = 1) = \frac{1}{4}$

Independent coin tosses



$\mathbb{P}(X_1 = 1) = \frac{1}{2}$ $\quad$ $\mathbb{P}(X_2 = 1) = \frac{1}{4}$ $\quad$ $\mathbb{P}(X_3 = 1) = \frac{3}{8}$

Independent coin tosses

Proposition [Boisson, Jamet, M. 2024]

$$\forall n \geq 2, \quad \mathbb{P}(X_n = 1) = \frac{1}{2}\left(1 - \frac{1}{2^{n-1}}\right).$$

Independent coin tosses

Proposition [Boisson, Jamet, M. 2024]

$$\forall n \geq 2, \quad \mathbb{P}(X_n = 1) = \frac{1}{2}\left(1 - \frac{1}{2^{n-1}}\right).$$

Proof: in order to sample X_n, it is sufficient to draw the values of $T_1, T_2, \ldots$ until knowing the block that contains X_n, and then to fill that block by an independent drawing.

Example for $n = 12$

We draw the beginning of $\mathbb{T}$ until identifying the block of X_{12}

- $\mathbb{T} = 2, 1, 2, 2, 1$
- $\mathbb{X} = \underbrace{22}_{2}\,\underbrace{11}_{2}\,\underbrace{2}_{1}\,\underbrace{2}_{1}\,\underbrace{11}_{2}\,\underbrace{??}_{2}\,\underbrace{?}_{1}\,\underbrace{?}_{1}$

Independent coin tosses

Proposition [Boisson, Jamet, M. 2024]

$$\forall n \geq 2, \quad \mathbb{P}(X_n = 1) = \frac{1}{2}\left(1 - \frac{1}{2^{n-1}}\right).$$

Proof: in order to sample X_n, it is sufficient to draw the values of $T_1, T_2, \ldots$ until knowing the block that contains X_n, and then to fill that block by an independent drawing.

Example for $n = 12$

We draw the beginning of $\mathbb{T}$ until identifying the block of X_{12}

- $\mathbb{T} = 2, 1, 2, 2, 1$
- $\mathbb{X} = \underbrace{22}_{2}\ \underbrace{11}_{2}\ \underbrace{2}_{1}\ \underbrace{2}_{1}\ \underbrace{11}_{2}\ \underbrace{??}_{2}\ \underbrace{?}_{1}\ \underbrace{?}_{1}$

It works **unless** $(T_1, T_2, \ldots, T_{n-2}, T_{n-1}) = (1, 1, \ldots, 1, 2)$ in which case the block of X_n is filled with 2's at the same time as we identify it: this is an event of probability $1/2^{n-1}$.

Independent drawings

Proposition [Boisson, Jamet, M. 2024]

If $\mathbb{T} \sim (p\delta_1 + (1-p)\delta_2)^{\otimes \mathbb{N}^\star}$ with $p \in]0,1[$, then

$$\begin{aligned} \forall n \geq 2, \qquad \mathbb{P}(X_n = 1) &= p(1 - p^{n-2}(1-p)) \\ &= p(1 - p^{n-2} + p^{n-1}). \end{aligned}$$

Independent drawings

Corollary [Boisson, Jamet, M. 2024]

If $\mathbb{T} \sim (p\delta_1 + (1-p)\delta_2)^{\otimes \mathbb{N}^\star}$ with $p \in]0,1[$, then

$$\lim_{n\to\infty} \frac{\mathbb{E}(|X_1 \dots X_n|_1)}{n} = p$$

Proof:

$$\begin{aligned}\mathbb{E}(|X_1 \dots X_n|_1) &= \sum_{k=1}^{n} \mathbb{E}(\mathbf{1}_{X_k=1}) \\ &= \sum_{k=1}^{n} \mathbb{P}(X_k = 1)\end{aligned}$$

Independent drawings

Theorem [Boisson, Jamet, M. 2024]

If $\mathbb{T} \sim (p\delta_1 + (1-p)\delta_2)^{\otimes \mathbb{N}^\star}$ with $p \in]0,1[$, then

$$\lim_{n\to\infty} \frac{|X_1 \dots X_n|_1}{n} = p \quad \text{a.s.}$$

We would like to apply the LLN to the $\mathbf{1}_{X_k=1}$ but they are not indep.

Independent drawings

Theorem [Boisson, Jamet, M. 2024]

If $\mathbb{T} \sim (p\delta_1 + (1-p)\delta_2)^{\otimes \mathbb{N}^\star}$ with $p \in]0,1[$, then

$$\lim_{n\to\infty} \frac{|X_1 \dots X_n|_1}{n} = p \quad \text{a.s.}$$

We would like to apply the LLN to the $\mathbf{1}_{X_k=1}$ but they are not indep.

LLN for weakly correlated variables [Lyons 1988]

Let $(Y_n)_{n\in\mathbb{N}^*}$ be a sequence of r.v. such that $\forall n \in \mathbb{N}^*$, $|Y_n| \leq 1$ a.s. and

$$\forall m, n \in \mathbb{N}^*,\ \mathbb{E}(Y_m Y_n) \leq \Phi(|n-m|),$$

where $\Phi \geq 0$ and $\sum_{n\geq 1} \frac{\Phi(n)}{n} < \infty$.
Then, $\frac{1}{N}\sum_{n=1}^{N} Y_n \underset{N\to\infty}{\longrightarrow} 0$ a.s.

Independent drawings

Lemma

If $\mathbb{T} \sim (p\delta_1 + (1-p)\delta_2)^{\otimes \mathbb{N}^\star}$ with $p \in]0,1[$, then for $n \geq 1$ and $k \geq 2$, we have

$$\mathbb{P}(X_n = 2 \text{ and } X_{n+k} = 1) = \mathbb{P}(X_n = 2) \times p.$$

Independent drawings

Lemma

If $\mathbb{T} \sim (p\delta_1 + (1-p)\delta_2)^{\otimes \mathbb{N}^\star}$ with $p \in]0,1[$, then for $n \geq 1$ and $k \geq 2$, we have

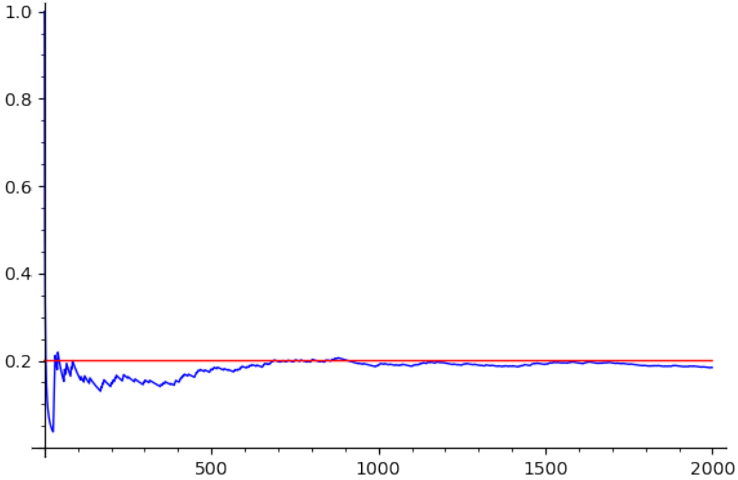
$$\mathbb{P}(X_n = 2 \text{ and } X_{n+k} = 1) = \mathbb{P}(X_n = 2) \times p.$$

Let $\tilde{X}_n = X_n - (2-p)$. We have $|\tilde{X}_n| \leq 1$ and $\lim_{n\to\infty} \mathbb{E}(\tilde{X}_n) = 0$.

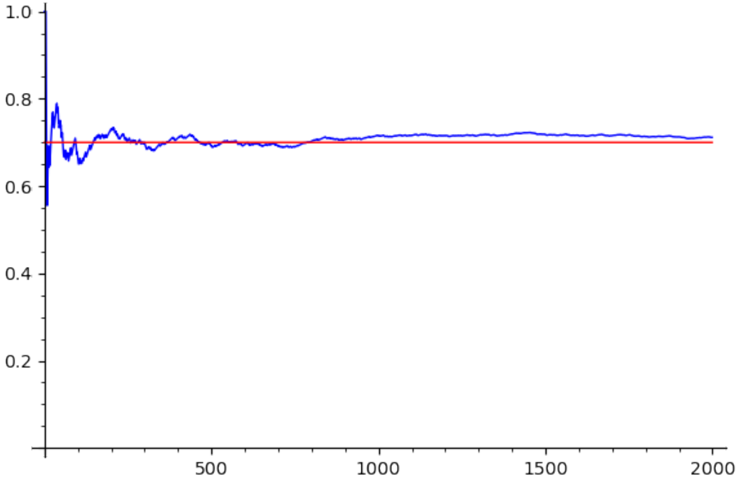
For $k \geq 2$, we have $\mathbb{E}(\tilde{X}_n \tilde{X}_{n+k}) \leq 0$.

We can apply the LLN of Lyons to the r.v. $\tilde{X}_n$.

Evolution of the frequency of 1's for an example of realisation with $p = 0.2$



Evolution of the frequency of 1's for an example of realisation with $p = 0.7$



Independent drawings

Proposition [Boisson, Jamet, M. 2024]

Let $a, b \in \mathbb{N}^\star$ be such that $1 < a < b$, and let $p \in]0,1[$.

1. If $\Sigma = \{1, a\}$ and $\mathbb{T} \sim (p\delta_1 + (1-p)\delta_a)^{\otimes \mathbb{N}^\star}$, then

$$\forall n \geq a, \quad \mathbb{P}(X_n = 1) = p\left(1 - p^{n-a} + p^{n-1}\right)$$

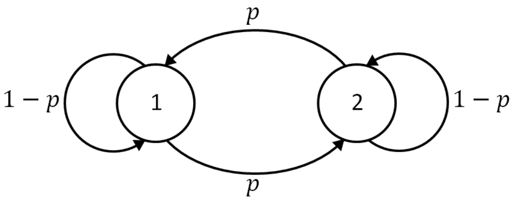
2. If $\Sigma = \{a, b\}$ and $\mathbb{T} \sim (p\delta_a + (1-p)\delta_b)^{\otimes \mathbb{N}^\star}$, then

$$\forall n \geq b+1, \quad \mathbb{P}(X_n = a) = p.$$

Markovian drawings

In order to get closer to the deterministic OK sequence, we consider a Markovian directing sequence: we choose a first value T_1, then:

$$\mathbb{P}(T_{n+1} = 2 | T_n = 1) = \mathbb{P}(T_{n+1} = 1 | T_n = 2) = p.$$



If $T_1 = 1$ and $p = 1$, we recover exactly the OK sequence!

For all $p \in]0,1[$, one can prove that:

$$\lim_{n\to\infty} \mathbb{P}(X_n = 1) = \frac{1}{2}.$$

Idea: with high probability, when we identify the block of X_n, there are many empty blocks in front of it. Now, the Markov chain is mixing, with stat. distrib. $\pi(1) = \pi(2) = 1/2$.

- $\mathbb{T} = 2, 1, 2, 2, 1$
- $\mathbb{X} = \underbrace{22}_{2}\ \underbrace{11}_{2}\ \underbrace{2}_{1}\ \underbrace{2}_{1}\ \underbrace{11}_{2}\ \underbrace{??}_{2}\ \underbrace{?}_{1}\ \underbrace{\color{red}{?}}_{1}$

Markovian drawings

For all $p \in]0,1[$, one can prove that:

$$\lim_{n\to\infty} \mathbb{P}(X_n = 1) = \frac{1}{2}.$$

Idea: with high probability, when we identify the block of X_n, there are many empty blocks in front of it. Now, the Markov chain is mixing, with stat. distrib. $\pi(1) = \pi(2) = 1/2$.

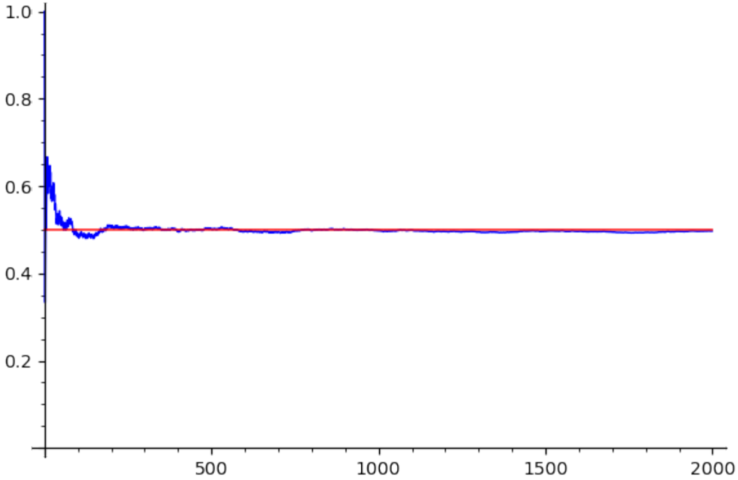
- $\mathbb{T} = 2, 1, 2, 2, 1$
- $\mathbb{X} = \underbrace{22}_{2}\ \underbrace{11}_{2}\ \underbrace{2}_{1}\ \underbrace{2}_{1}\ \underbrace{11}_{2}\ \underbrace{??}_{2}\ \underbrace{?}_{1}\ \underbrace{\color{red}{?}}_{1}$

Consequently,

$$\lim_{n\to\infty} \frac{\mathbb{E}(|X_1 \dots X_n|_1)}{n} = \frac{1}{2}.$$

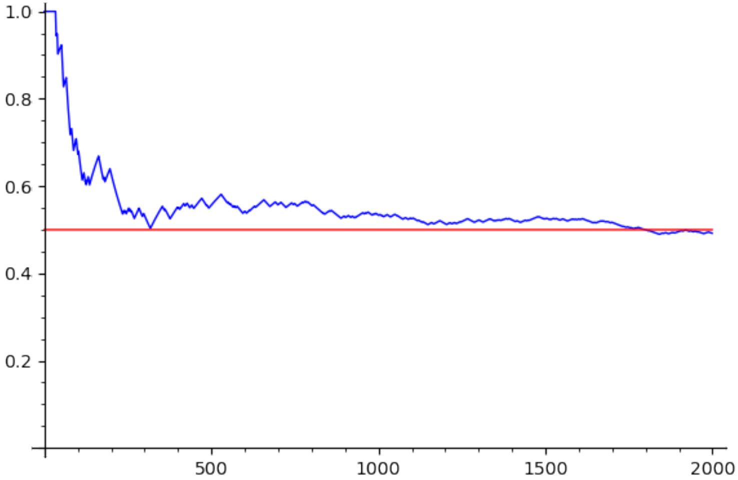
Markovian drawings

Evolution of the frequency of 1's for an example of realisation with $p = 0.9$



Markovian drawings

Evolution of the frequency of 1's for an example of realisation with $p = 0.2$



We conjecture that the limit is also almost sure.

Question: can we let p tend to 1 and thus solve Keane's conjecture?

Markovian drawings on $\{1,3\}$

A little detour through the case of the alphabet $\{1,3\}$:

- For the directing sequence $T = 1,3,1,3,1,3,1,3,\ldots$, we know that the frequency of 1's converges to an algebraic number $f_1 \approx 0.3972$.

$$\mathcal{O}_{1,3} = \underbrace{1}_{1}\,\underbrace{333}_{3}\,\underbrace{111}_{3}\,\underbrace{333}_{3}\,\underbrace{1}_{1}\,\underbrace{3}_{1}\,\underbrace{1}_{1}\,\underbrace{333}_{3}\,\underbrace{111}_{3}\,\underbrace{333}_{3}\,\ldots$$

Markovian drawings on $\{1,3\}$

A little detour through the case of the alphabet $\{1,3\}$:

- For the directing sequence $T = 1,3,1,3,1,3,1,3,\ldots$, we know that the frequency of 1's converges to an algebraic number $f_1 \approx 0.3972$.

$$\mathcal{O}_{1,3} = \underbrace{1}_{1}\underbrace{333}_{3}\underbrace{111}_{3}\underbrace{333}_{3}\underbrace{1}_{1}\underbrace{3}_{1}\underbrace{1}_{1}\underbrace{333}_{3}\underbrace{111}_{3}\underbrace{333}_{3}\ldots$$

- For the Markovian case on $\{1,3\}$, the same arguments as above prove that for all $p \in]0,1[$,

$$\lim_{n\to\infty} \mathbb{P}(X_n = 1) = \frac{1}{2}.$$

Markovian drawings on $\{1, 3\}$

A little detour through the case of the alphabet $\{1, 3\}$:

- For the directing sequence $T = 1, 3, 1, 3, 1, 3, 1, 3, \ldots$, we know that the frequency of 1's converges to an algebraic number $f_1 \approx 0.3972$.

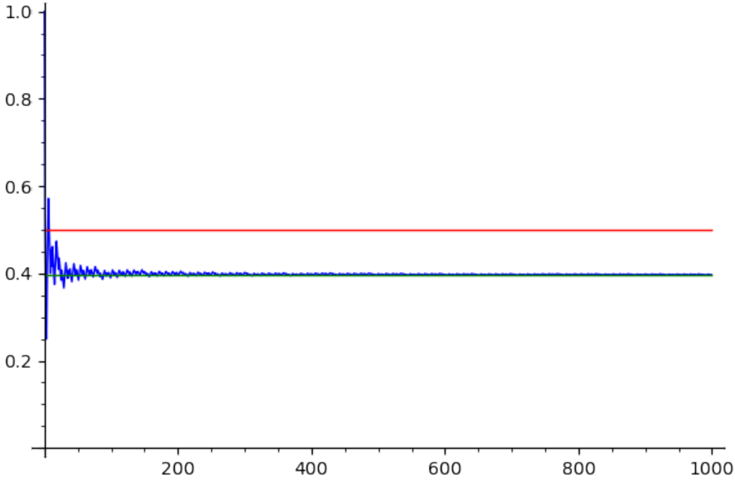
$$\mathcal{O}_{1,3} = \underbrace{1}_{1}\underbrace{333}_{3}\underbrace{111}_{3}\underbrace{333}_{3}\underbrace{1}_{1}\underbrace{3}_{1}\underbrace{1}_{1}\underbrace{333}_{3}\underbrace{111}_{3}\underbrace{333}_{3}\ldots$$

- For the Markovian case on $\{1, 3\}$, the same arguments as above prove that for all $p \in]0, 1[$,

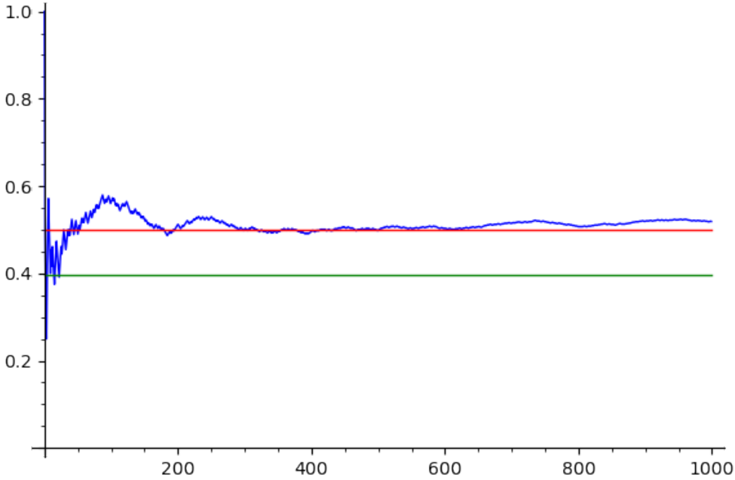
$$\lim_{n\to\infty} \mathbb{P}(X_n = 1) = \frac{1}{2}.$$

Conclusion: we cannot let p tend to 1...

Evolution of the frequency of 1's
with $p = 1$

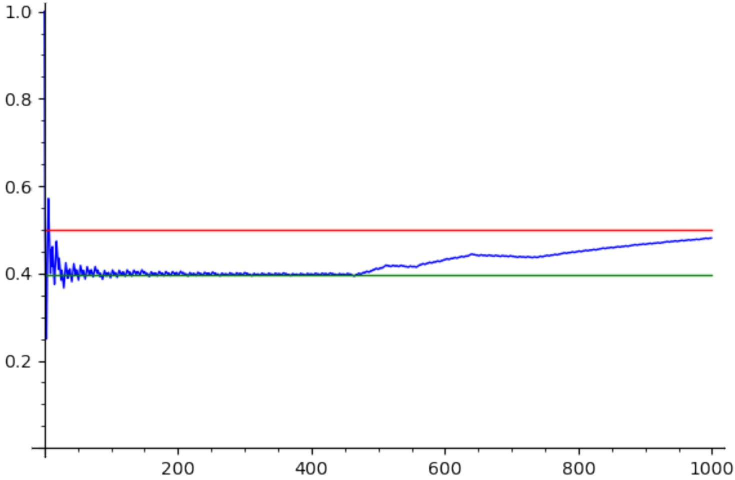


Evolution of the frequency of 1's for an example of realisation with $p = 0.9$



Markovian drawings on $\{1, 3\}$

Evolution of the frequency of 1's for an example of realisation with $p = 0.99$



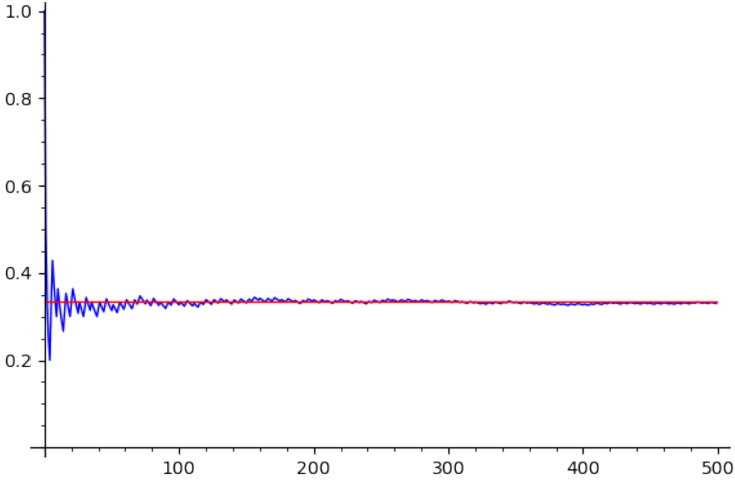
Conservation of the frequency

Question: on $\{1, 2\}$, is the frequency of 1's in the directing sequence $\mathbb{T}$ always the same as in the resulting sequence $\mathbb{X}$?

Outline

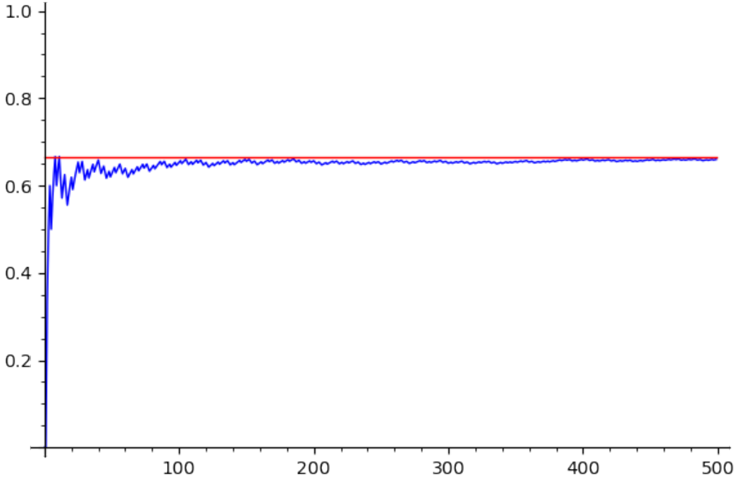
1. The Oldenburger-Kolakoski sequence
2. Self-descriptive sequences directed by a sequence
   - Random directing sequences
   - Determinstic directing sequences
3. Smooth sequences

Evolution of the frequency of 1's for $T = (122)^{\infty}$

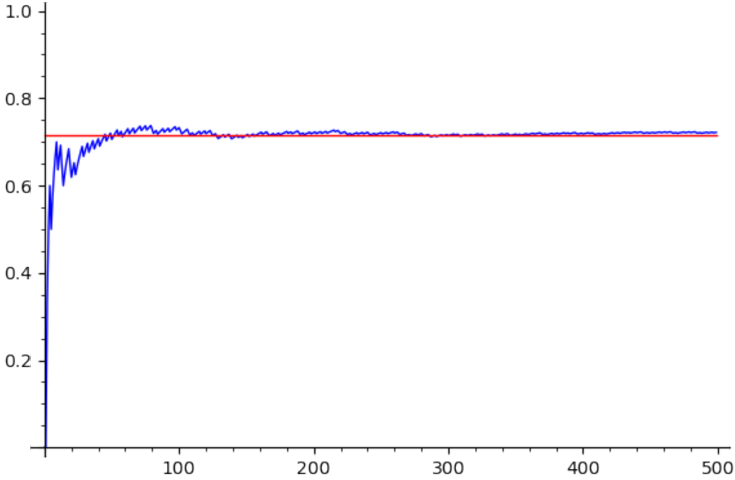


Conservation of the frequency

Evolution of the frequency of 1's for $T = (211)^\infty$



Evolution of the frequency of 1's for $T = (2112111)^\infty$



Non-conservation of the frequency

We can construct an example of sequence for which the frequency is not preserved.

Idea: we start by a 2, then we fill

- the next blocks of size 2 of $\mathcal{O}_T$ by 1's,
- blocks of size 1's by 1's and 2's, alternatively.

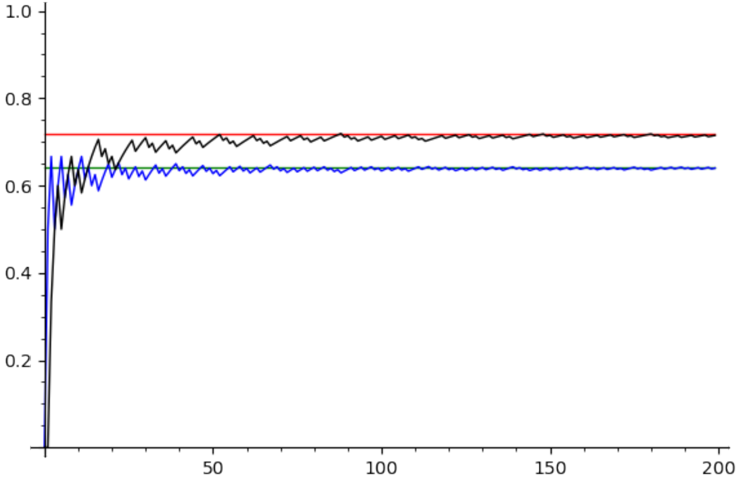
We thus build simultaneously the directing sequence and the resulting sequence, in such a way that there are more 1's in $\mathcal{O}_T$ than in T.

$$T = 2, 1, 1, 2, \ldots$$

$$\mathcal{O}_T = \underbrace{22}_{2}\ \underbrace{11}_{2}\ \underbrace{1}_{1}\ \underbrace{2}_{1}\ \underbrace{?}_{1}\ \underbrace{??}_{2} \ldots$$

Example of non-conservation of the frequency

Evolution of the frequency of 1's for $\mathcal{O}_T$ (black) and T (blue)



Example of non-conservation of the frequency

Theorem [Boisson, Jamet, M. 2024]

If the frequencies f_1^T et $f_1^{\mathcal{O}}$ exist, then:

$$f_1^T = \frac{1+\sqrt{17}}{8} = 0.640\ldots, \qquad f_1^{\mathcal{O}} = \frac{7-\sqrt{17}}{4} = 0.719\ldots$$

$$\mathcal{O}^{(4)} = \underbracket{22}\underbracket{11}\underbracket{1}\underbracket{2}\underbracket{?}\underbracket{??} \qquad \mathrm{T}^{(4)} = (2,1,1,2)$$

Example of non-conservation of the frequency

Theorem [Boisson, Jamet, M. 2024]

If the frequencies f_1^T et $f_1^{\mathcal{O}}$ exist, then:

$$f_1^T = \frac{1+\sqrt{17}}{8} = 0.640\ldots, \qquad f_1^{\mathcal{O}} = \frac{7-\sqrt{17}}{4} = 0.719\ldots$$

$$\mathcal{O}^{(4)} = \underbracket{22}\underbracket{11}\underbracket{1}\underbracket{2}\underbracket{?}\underbracket{??} \qquad \mathrm{T}^{(4)} = (2,1,1,2)$$

- $|\mathcal{O}^{(n)}|_2 \approx |\mathrm{T}^{(n)}|_2$
- $|\mathcal{O}^{(n)}|_1 \approx |\mathrm{T}^{(n)}|_2 + 2(|\mathrm{T}^{(n)}|_1 - |\mathrm{T}^{(n)}|_2)$

Consequence: if f_1^T exists, then $f_1^{\mathcal{O}}$ also exists, and

$$f_1^{\mathcal{O}} = \frac{3f_1^T - 1}{2f_1^T}.$$

Example of non-conservation of the frequency

Theorem [Boisson, Jamet, M. 2024]

If the frequencies f_1^T et $f_1^{\mathcal{O}}$ exist, then:

$$f_1^T = \frac{1+\sqrt{17}}{8} = 0.640\ldots, \qquad f_1^{\mathcal{O}} = \frac{7-\sqrt{17}}{4} = 0.719\ldots$$

$$\mathcal{O}^{(4)} = \sqcup 22 \sqcup 11 \sqcup 1 \sqcup 2 \sqcup ? \sqcup ?? \sqcup \qquad \mathrm{T}^{(4)} = (2,1,1,2)$$

- $|\mathcal{O}^{(n)}|_2 \approx |\mathrm{T}^{(n)}|_2$
- $|\mathcal{O}^{(n)}|_1 \approx |\mathrm{T}^{(n)}|_2 + 2(|\mathrm{T}^{(n)}|_1 - |\mathrm{T}^{(n)}|_2)$

Consequence: if f_1^T exists, then $f_1^{\mathcal{O}}$ also exists, and $f_1^{\mathcal{O}} = \dfrac{3f_1^T - 1}{2f_1^T}$.

Furthermore, $f_1^T = f_2^{\mathcal{O}} + \dfrac{1}{2} f_1^{\mathcal{O}}$.

Example of non-conservation of the frequency

Theorem [Akiyama, Jamet, M., Trân Cong 2024]

The frequencies f_1^T et $f_1^{\mathcal{O}}$ do exist.

$\mathcal{O}_T$ is the fixed point of a rewriting rule: $\mathcal{O}_T = \lim_{n\to\infty} \sigma^n(2)$,

$$\text{with:} \qquad \sigma : \begin{array}{ccl} 1 & \mapsto & 1, 2, 1, 2, \cdots \\ 2 & \mapsto & 22, 11, 11, 11, \cdots \end{array}$$

$$v_n = \begin{pmatrix} |\sigma^n(2)|_1 \\ |\sigma^n(2)|_2 \end{pmatrix}$$

$$v_{n+1} = \begin{pmatrix} 1/2 & 2 \\ 1/2 & 0 \end{pmatrix} \cdot v_n + e_n, \quad \text{with} \quad e_n \in \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} \right\}$$

$$\lim_{n\to\infty} \frac{(|\sigma^n(2)|_1, |\sigma^n(2)|_2)}{|\sigma^n(2)|} = (r_1, r_2)$$

Existence of the frequencies along an extracted subsequence.
Conclusion with an *accordion argument*.

Extending the counter-example

K_{T_1,T_2} is a self-descriptive sequence such that:

- blocks of length 1 are filled with T_1's letters,
- blocks of length 2 are filled with T_2's letters,

where T_1 and T_2 are two sequences on $\{1,2\}$.

For instance, if $T_1 = (112)^\infty$ and $T_2 = (12)^\infty$:

$K_{T_1,T_2} = \underbrace{22}_{2}\ \underbrace{1\ 1}_{2}\ \underbrace{1}_{1}\ \underbrace{1}_{1}\ \underbrace{2}_{1}\ \underbrace{1}_{1}\ \underbrace{2\ 2}_{2}\ \underbrace{1}_{1}\ \underbrace{1\ 1}_{2}\ \underbrace{2\ 2}_{2}\ \underbrace{2}_{1}\ \underbrace{?}_{1}\ \underbrace{?}_{1}\ \underbrace{??}_{2}\ \underbrace{??}_{2}\ \ldots$

$T = 21112121122\ldots$

Extending the counter-example

K_{T_1,T_2} is a self-descriptive sequence such that:

- blocks of length 1 are filled with T_1's letters,
- blocks of length 2 are filled with T_2's letters,

where T_1 and T_2 are two sequences on $\{1,2\}$.

For instance, if $T_1 = (112)^\infty$ and $T_2 = (12)^\infty$:

$K_{T_1,T_2} = \underbrace{22}_{2}\underbrace{1\ 1}_{2}\underbrace{1}_{1}\underbrace{1}_{1}\underbrace{2}_{1}\underbrace{1}_{1}\underbrace{2\ 2}_{2}\underbrace{1}_{1}\underbrace{1\ 1}_{2}\underbrace{2\ 2}_{2}\underbrace{2}_{1}\underbrace{?}_{1}\underbrace{?}_{1}\underbrace{??}_{2}\underbrace{??}_{2}\ldots$

$T = 2111212112 2\ldots$

Theorem [Akyiama, Jamet, M., Trân Cong 2024]

If T_1 and T_2 are periodic sequences and if

$$M = \begin{pmatrix} f_1^{T_1} & 2f_1^{T_2} \\ f_2^{T_1} & 2f_2^{T_2} \end{pmatrix}$$

is primitive, then the frequencies of K_{T_1,T_2} exist and are given by the right Perron eigenvector of M.

Outline

Smooth sequences

A sequence $x \in \Sigma^{\mathbb{N}}$ is **smooth** if $\forall n \geq 0, \Delta^n(x) \in \Sigma^{\mathbb{N}}$.

$$\begin{aligned}
x &= 221221121121221121122 1\cdots \\
\Delta(x) &= 212212112212 2\cdots \\
\Delta^2(x) &= 112112212\cdots \\
\Delta^3(x) &= 21221\cdots \\
\Delta^4(x) &= 112\cdots \\
\Delta^5(x) &= 2\cdots
\end{aligned}$$

Smooth sequences

A sequence $x \in \Sigma^{\mathbb{N}}$ is **smooth** if $\forall n \geq 0, \Delta^n(x) \in \Sigma^{\mathbb{N}}$.

$$
\begin{aligned}
x &= 2212211211212211211221\cdots \\
\Delta(x) &= 212212112212\cdots \\
\Delta^2(x) &= 112112212\cdots \\
\Delta^3(x) &= 21221\cdots \\
\Delta^4(x) &= 112\cdots \\
\Delta^5(x) &= 2\cdots
\end{aligned}
$$

The set $\mathcal{S}_{1,2}$ of smooth sequences is homeomorphic to $\{1,2\}^{\mathbb{N}}$.

$$
\begin{aligned}
\varphi : \mathcal{S}_{1,2} &\to \{1,2\}^{\mathbb{N}} \\
x &\mapsto (\Delta^n(x)_0)_{n\geq 0}
\end{aligned}
$$

Smooth sequences

A sequence $x \in \Sigma^{\mathbb{N}}$ is **smooth** if $\forall n \geq 0, \Delta^n(x) \in \Sigma^{\mathbb{N}}$.

$$
\begin{aligned}
x &= 22122112112122112112121\cdots \\
\Delta(x) &= 212212112212\cdots \\
\Delta^2(x) &= 112112212\cdots \\
\Delta^3(x) &= 21221\cdots \\
\Delta^4(x) &= 112\cdots \\
\Delta^5(x) &= 2\cdots
\end{aligned}
$$

The set $\mathcal{S}_{1,2}$ of smooth sequences is homeomorphic to $\{1,2\}^{\mathbb{N}}$.

$$
\begin{aligned}
\varphi : \mathcal{S}_{1,2} &\to \{1,2\}^{\mathbb{N}} \\
x &\mapsto (\Delta^n(x)_0)_{n\geq 0}
\end{aligned}
$$

Conjecture: for any $x \in \mathcal{S}_{1,2}$, the frequencies of 1's and 2's in x exist and equal $1/2$.

$$
\begin{aligned}
x &= {\color{red}3}131113331113131113111313111333111\cdots \\
\Delta(x) &= {\color{red}1}11333111313111333111311133311131\cdots \\
\Delta^2(x) &= {\color{red}3}3311133313331\cdots \\
\Delta^3(x) &= {\color{red}3}3313\cdots \\
\Delta^4(x) &= {\color{red}3}1\cdots \\
\Delta^5(x) &= {\color{red}1}\cdots
\end{aligned}
$$

Question: what can be said about the frequencies of 1's and 3's in smooth sequences over $\{1,3\}$?

Smooth sequences on $\{1,3\}$

$$
\begin{aligned}
x &= 313 \cdot 111333111 \cdot 313 \cdot 111 \cdot 3 \cdot 111 \cdot 313 \cdot 111333111 \cdots \\
\Delta(x) &= 111 \cdot 333 \cdot 111 \cdot 3 \cdot 1 \cdot 3 \cdot 111 \cdot 333 \cdots
\end{aligned}
$$

Four blocks: $\mathsf{A} = 1, \mathsf{B} = 3, \mathsf{C} = 111, \mathsf{D} = 333$.

The two primitives of a smooth sequence are given by the two substitions:

$$
\varphi_0 : \begin{cases} \mathsf{A} & \mapsto \quad \mathsf{A} \\ \mathsf{B} & \mapsto \quad \mathsf{D} \\ \mathsf{C} & \mapsto \quad \mathsf{ABA} \\ \mathsf{D} & \mapsto \quad \mathsf{DCD} \end{cases} \qquad \varphi_1 : \begin{cases} \mathsf{A} & \mapsto \quad \mathsf{B} \\ \mathsf{B} & \mapsto \quad \mathsf{C} \\ \mathsf{C} & \mapsto \quad \mathsf{BAB} \\ \mathsf{D} & \mapsto \quad \mathsf{CDC} \end{cases}
$$

$$\begin{aligned} x &= 313\cdot 111333111\cdot 313\cdot {\color{blue}111}\cdot 3\cdot {\color{blue}111}\cdot 313\cdot 111333111\cdots \\ \Delta(x) &= 111\cdot 333\cdot 111\cdot {\color{blue}3}\cdot 1\cdot {\color{blue}3}\cdot 111\cdot 333\cdots \end{aligned}$$

Four blocks: $\mathsf{A}=1, \mathsf{B}=3, \mathsf{C}=111, \mathsf{D}=333$.

The two primitives of a smooth sequence are given by the two substitions:

$$\varphi_0 : \begin{cases} \mathsf{A} & \mapsto \mathsf{A} \\ \mathsf{B} & \mapsto \mathsf{D} \\ \mathsf{C} & \mapsto \mathsf{ABA} \\ \mathsf{D} & \mapsto \mathsf{DCD} \end{cases} \qquad \varphi_1 : \begin{cases} \mathsf{A} & \mapsto \mathsf{B} \\ \mathsf{B} & \mapsto \mathsf{C} \\ \mathsf{C} & \mapsto \mathsf{BAB} \\ \mathsf{D} & \mapsto \mathsf{CDC} \end{cases}$$

Smooth sequences on $\{1,3\}$

$$
\begin{aligned}
x &= \overset{\mathsf{BAB}}{313}\cdot\overset{\mathsf{C}}{111}\overset{\mathsf{D}}{333}\overset{\mathsf{C}}{111}\cdot\overset{\mathsf{BAB}}{313}\cdot\overset{\mathsf{C}}{111}\cdot\overset{\mathsf{B}}{3}\cdot\overset{\mathsf{C}}{111}\cdot\overset{\mathsf{BAB}}{313}\cdot\overset{\mathsf{C}}{111}\overset{\mathsf{D}}{333}\overset{C}{111}\cdots\\
\Delta(x) &= \underset{\mathsf{C}}{111}\cdot\underset{\mathsf{D}}{333}\cdot\underset{\mathsf{C}}{111}\cdot\underset{\mathsf{B}}{3}\cdot\underset{\mathsf{A}}{1}\cdot\underset{\mathsf{B}}{3}\cdot\underset{\mathsf{C}}{111}\cdot\underset{\mathsf{D}}{333}\cdots
\end{aligned}
$$

Four blocks: $\mathsf{A}=1, \mathsf{B}=3, \mathsf{C}=111, \mathsf{D}=333$.

The two primitives of a smooth sequence are given by the two substitions:

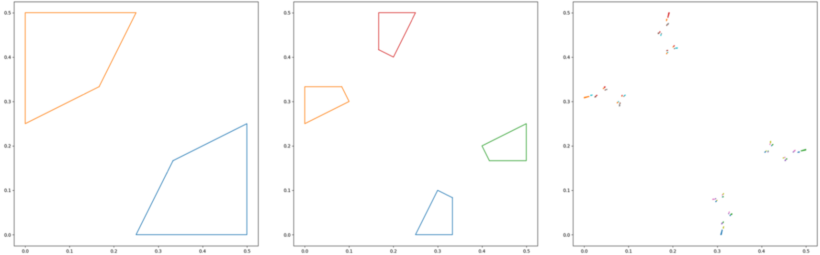
$$
\varphi_0:\begin{cases}\mathsf{A} & \mapsto \ \mathsf{A}\\ \mathsf{B} & \mapsto \ \mathsf{D}\\ \mathsf{C} & \mapsto \ \mathsf{ABA}\\ \mathsf{D} & \mapsto \ \mathsf{DCD}\end{cases}
\qquad
\varphi_1:\begin{cases}\mathsf{A} & \mapsto \ \mathsf{B}\\ \mathsf{B} & \mapsto \ \mathsf{C}\\ \mathsf{C} & \mapsto \ \mathsf{BAB}\\ \mathsf{D} & \mapsto \ \mathsf{CDC}\end{cases}
$$

Smooth sequences on $\{1, 3\}$

If $\Delta(x)$ has frequencies (a, b) of blocks A and B, then the frequencies of its two primitives are given by:

$$h_0(a, b) = \frac{1}{3 - 2(a + b)} \left(1 - a, \frac{1}{2} - a\right)$$

$$h_1(a, b) = \frac{1}{3 - 2(a + b)} \left(\frac{1}{2} - a, 1 - a\right)$$

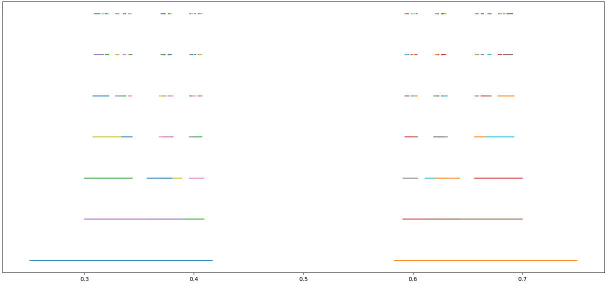


Smooth sequences on $\{1, 3\}$

The homographies h_0 and h_1 are contracting.
For any column sequence, there is only one possible value for the frequencies of A and B in the "top" sequence x.

Theorem [Jamet, M., Poirier, de la Rue 2026]

For any $x \in \mathcal{S}_{1,3}$, the frequencies of 1's and 3's in x exist.
Furthermore, these frequencies can be described using an iterated function system.



Smooth sequences on $\{1, 3\}$

Theorem [Jamet, M., Poirier, de la Rue 2026]

For any $\tau \in \{0,1\}^{\mathbb{N}}$, the subshift (X_τ, σ) is uniquely ergodic. In particular, for any smooth word $x \in \mathcal{S}_{1,3}$, and any pattern p, the frequency of the pattern p in the sequence x is well-defined, and uniquely determined by the sequence τ.

Conclusion and future work

Conclusion

- For i.i.d. drawings of T, $\mathcal{O}_T$ and T have the same frequencies.
- For Markov chain drawings of T, the frequencies of $\mathcal{O}_T$ and T are $1/2$ (in average), but the deterministic case cannot be obtained by making $p \to 1$.
- There exist sequences T and $\mathcal{O}_T$ with different frequencies, explicitly computed.
- Smooth sequences on $\{1, 3\}$ all have frequencies, explicitly described.

Conclusion and future work

Conclusion

- For i.i.d. drawings of T, $\mathcal{O}_T$ and T have the same frequencies.
- For Markov chain drawings of T, the frequencies of $\mathcal{O}_T$ and T are $1/2$ (in average), but the deterministic case cannot be obtained by making $p \to 1$.
- There exist sequences T and $\mathcal{O}_T$ with different frequencies, explicitly computed.
- Smooth sequences on $\{1, 3\}$ all have frequencies, explicitly described.

Perspectives

- Relaxing assumptions on T_1 and T_2.
- Is the frequency conserved for any periodic directing sequence T?
- Keane's conjecture is still open. More generally, we conjecture that the freq. exist and equal $1/2$ for all smooth sequences on $\{1, 2\}$.