LAPLACE, CAUCHY & EARLY ANALYTIC COMBINATORICS

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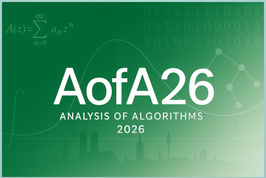
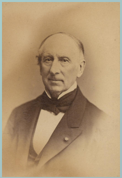
(with LI Chong-Yi@*AS* & Vytas Zacharovas@*Vilnius U.*)

P.-S. Laplace
1749–1827



AofA
June 24, 2026



A.-L. Cauchy
1789–1857

Structures

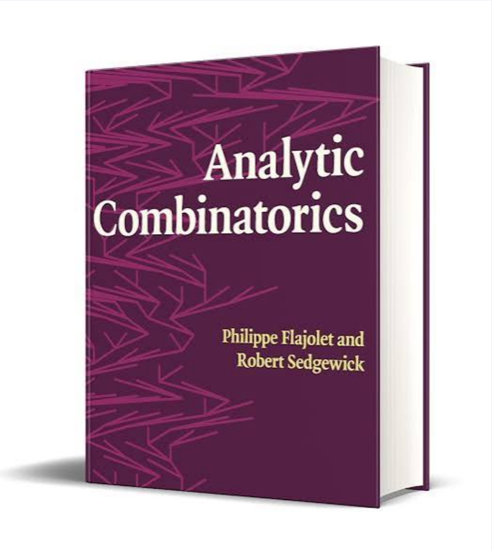
Symbolic calculus

Generating function

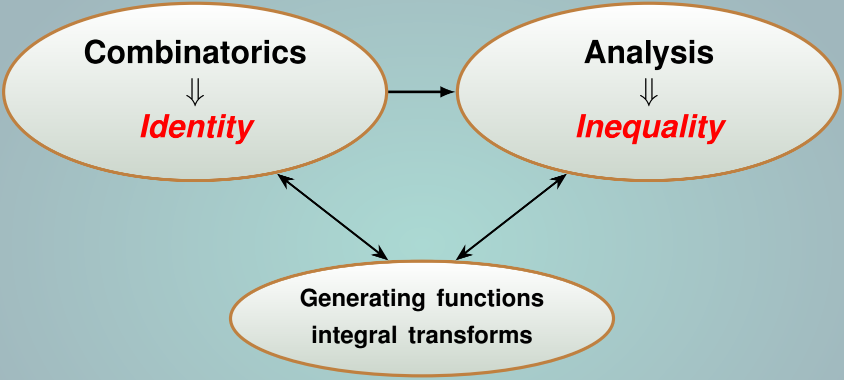
$$f(z) = \sum_{n \geqslant 0} a_n z^n$$

Asymptotics
(Limit laws)

$$a_n = [z^n] f(z) \sim ?$$



ANALYTIC COMBINATORICS



Finite	$\longrightarrow$	*Infinite*	*Formal*	$\longrightarrow$	*Analytic*
Discrete	$\longrightarrow$	*Continuous*	*Exact*	$\longrightarrow$	*Approximate*

ANALYTIC POETRY

William Wordsworth (1770–1850):

***Ten thousand** saw I at a glance,*
Tossing their heads in sprightly dance.

Literal
⇓
Equality

→

Hyperbolic
⇓
Inequality

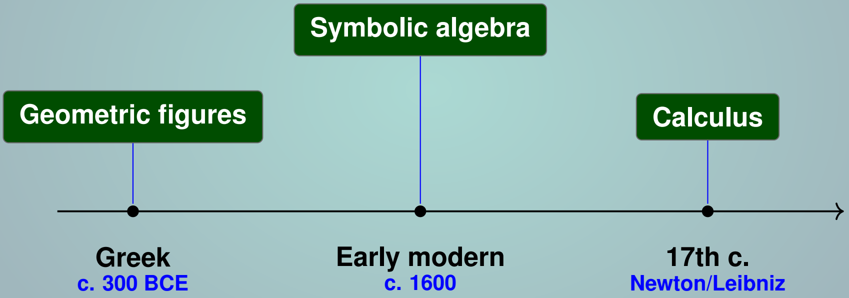
李白（762CE）：白髮三千丈

(White hair, 3000 feet long)

ANALYSIS IN MATHEMATICS

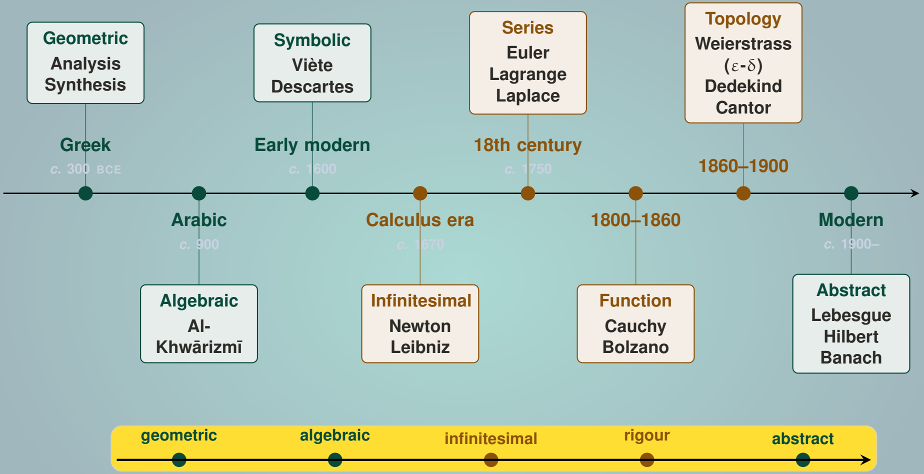
Ancient Greek, *analyse* = ἀναλύω *(analýō): resolve by parts*

(decompose, examine, synthesize)



Aim to resolve similar, tools/techniques evolving . . .

EVOLUTION OF ANALYTIC TOOLS & NOTIONS



Lagrange (1788): On ne trouvera point de Figures dans cet Ouvrage.

Mécanique analytique: mechanics reduced to algebraic operations

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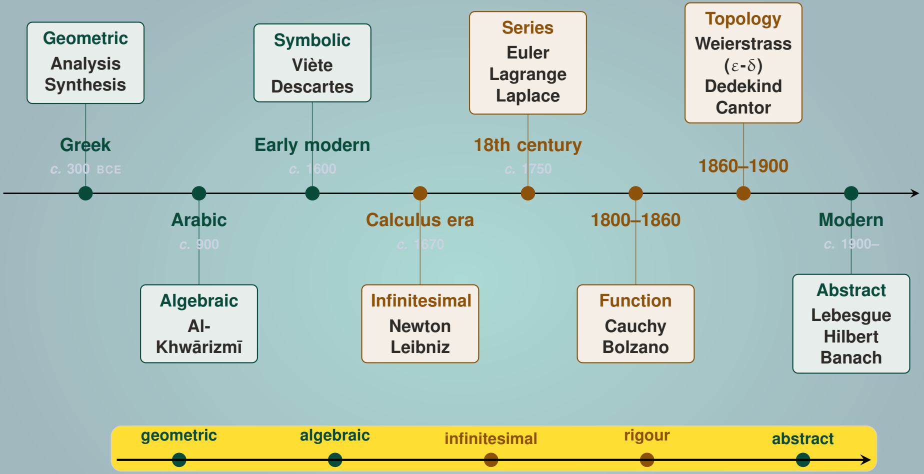

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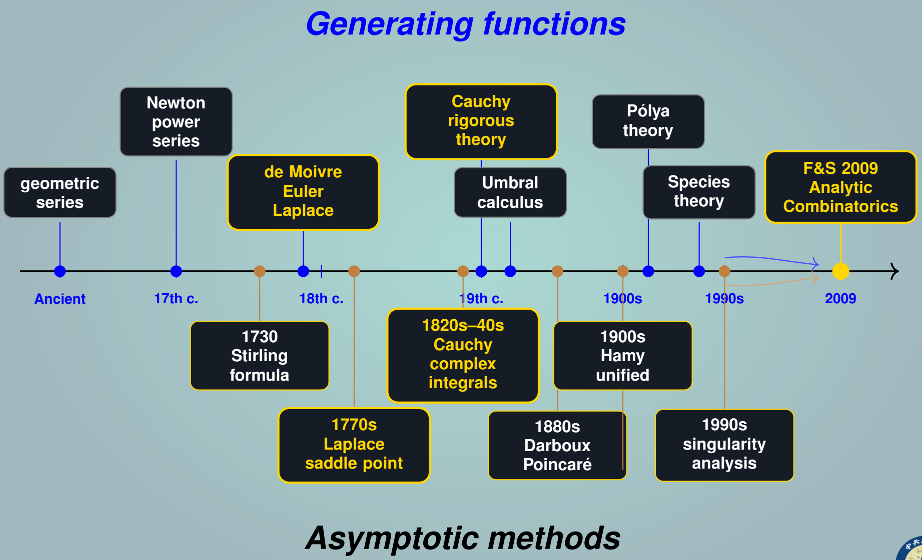
TIMELINE OF ANALYTIC COMBINATORICS



EARLY SERIES EXPANSIONS

From geometric sums to binomial arrays

- **Euclid (c. 300 BCE, *Elements*): finite geometric progression**

$$1 + x + x^2 + \cdots + x^{n-1} = \tfrac{1-x^n}{1-x}$$

- **Archimedes (c. 250 BCE): geometric exhaustion, e.g.**

$$1 + \tfrac{1}{4} + \tfrac{1}{4^2} + \cdots = \tfrac{4}{3}$$

- **Pingala (c. 3rd–2nd c. BCE): binomial coefficients**
- **al-Karajī (c. 1000): Pascal triangle and integer binomial expansion**

$$(1 + x)^n = \sum_{0 \leqslant j \leqslant n} \binom{n}{j} x^j$$

- **Kerala School, Madhava tradition (c. 1400): sine, cosine, arctangent, geometric series**
- **Torricelli (mid-14th c.): general infinite geometric series**

$$1 + x + x^2 + \cdots = \tfrac{1}{1-x}$$

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BIRTH OF CALCULUS: MORE SERIES EXPANSIONS

The 17th century: power series become a method

- **Newton (1665–66; *Epistola prior*, 1676):** $(1+x)^r = \sum_{j\geqslant 0} \binom{r}{j} x^j$
- **Mercator (1668):** $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$
- **Newton, *De analysi* (1669):** $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, **&** $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$
- **Gregory (1670–71): power series for** $\tan x$**,** $\sec x$**,** $\arctan x$ $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$, **&** $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots$
- **Gregory (1671), Leibniz (1673): European rediscovery of** $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

Next stage (the 18th century): Taylor series & its use

DE MOIVRE (1667–1754)

Newton: he [de M.] knows these things better than I do

A pioneer of analytic combinatorics?

- **1667: born in France (Vitry-le-François)**
- **1687: exile to London**
- **1697: elected member FRS**
- **Math consultant in Slaughter's Coffee House (for probability, annuities, gambling problems, & risk calculation)**
- **1754: died in London**

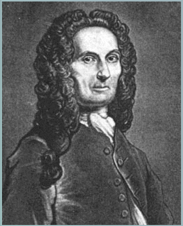
- **1704:** ***Animadversiones in D. Georgii Cheynaei Tractatum de Fluxionum Methodo Inversa***
- **1718:** ***Doctrine of Chances***
- **1725:** ***A Treatise of Annuities upon Lives***
- **1730:** ***Miscellanea Analytica***

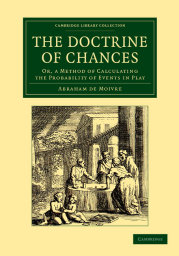
Slept* 15 *minutes more every day* $\Rightarrow$ *predicted his own death.

DE MOIVRE & ANALYTIC COMBINATORICS

A pioneer of analytic combinatorics

- **Expansions of power series**
  - **1697:** $(a_1 z + a_2 z^2 + \cdots)^k = \sum_{n \geqslant k} A_{n,k} z^n$
  - **1698: series reversion** $y = a_1 x + \cdots \Rightarrow x = \alpha_1 y + \cdots$
- **Series expansions as generating functions for computing probabilities**
  - **1711: *De Mensura Sortis***
  - **1718/1738/1756: *The Doctrine of Chances***
- **Partial fraction decomposition (solving linear recurrence)**
  - **1722:** $A(z) = \frac{P(z)}{Q(z)} \rightsquigarrow [z^n]A(z) = \sum_{1 \leqslant j \leqslant d} C_j \alpha_j^n$
  - **If** n **is large, then** $[z^n]A(z) \approx$ **the largest term**
- $(\cos t + i \sin t)^n = \cos nt + i \sin nt$





DE MOIVRE & ANALYTIC COMBINATORICS

The first normal limit law

- **Abraham de Moivre (*Miscellanea Analytica*, 1730):**

$$\frac{1}{4^n}\binom{2n}{n} \approx 2.168\,\frac{(1-1/(2n))^{2n}}{\sqrt{2n-1}} \approx \frac{2.168}{e\sqrt{2n}}$$

  He constructed a table of $\log n!$ **for** $n = 10, 20, \ldots, 900$ **(to 14 digits each).**

- **Stirling (*Methodus Differentialis*, 1730): numerical errors in de Moivre's table past the fifth decimal:** $z = n + \frac{1}{2}$

$$\log n! \approx z \log z - z + \log\sqrt{2\pi} - \frac{1}{2\cdot 12z} + \frac{7}{8\cdot 360z^3} - \frac{31}{32\cdot 1260z^5} - \cdots$$

- **de Moivre (1730): coefficients connected to Bernoulli #s**

$$\log n! \approx (n+\tfrac{1}{2})\log n - n + 1 + \sum_{k\geqslant 1}\frac{B_{2k}}{2k(2k-1)}\left(n^{1-2k}-1\right)$$

De Moivre (1730): MISCELLANEA ANALYTICA

n	$\log(n!)$ (1730)	$\log(n!)$ (1756)	(1730) − (1756)
10	6.55977 303287678	6.55976 303287678	$+10^{-5}$
20	18.38613 461687770	18.38612 461687770	$+10^{-5}$
30	32.42367 007492572	32.42366 007492572	$+10^{-5}$
40	47.91165 506815591	47.91164 506815591	$+10^{-5}$
50	64.48308 487247209	64.48307 487247209	$+10^{-5}$
60	81.92018 484939024	81.92017 484939024	$+10^{-5}$
70	100.07841 503568004	100.07840 503568004	$+10^{-5}$
80	118.85473 71 2249966	118.85472 77 2249966	$+10^{-5} - 6 \cdot 10^{-7}$
90	138.17194 51 9001086	138.17193 57 9001086	$+10^{-5} - 6 \cdot 10^{-7}$
100	157.97001 30 5471585	157.97000 36 5471585	$+10^{-5} - 6 \cdot 10^{-7}$
110	178.20092 70 4487008	178.20091 76 4487008	$+10^{-5} - 6 \cdot 10^{-7}$
120	198.82540 32 4721977	198.82539 38 4721977	$+10^{-5} - 6 \cdot 10^{-7}$
130	219.81070 25 5614815	219.81069 31 5614815	$+10^{-5} - 6 \cdot 10^{-7}$
140	241.12911 93 8869689	241.12910 99 8869689	$+10^{-5} - 6 \cdot 10^{-7}$
150	262.75690 28 1092616	262.75689 34 1092616	$+10^{-5} - 6 \cdot 10^{-7}$
160	284.67346 56 4068298	284.67345 62 4068298	$+10^{-5} - 6 \cdot 10^{-7}$
170	306.66079 13 9482847	306.66078 19 9482847	$+10^{-5} - 6 \cdot 10^{-7}$
180	329.30298 08 238 9393	329.30297 14 247 9393	$+10^{-5} - 6 \cdot 10^{-7} - 9 \cdot 10^{-9}$
190	351.98589 92 366 3535	351.98588 98 339 3535	$+10^{-5} - 6 \cdot 10^{-7} + 27 \cdot 10^{-10}$
200	374.89689 80 427 4044	374.89688 86 400 4044	$+10^{-5} - 6 \cdot 10^{-7} + 27 \cdot 10^{-10}$

Gélinas (2017)

GENERATING FUNCTIONS: EULER & LAGRANGE

Systematic manipulation of generating functions

- **Euler (1707–1783), *functional encoding*: integer partitions**

$$\prod_{m\geqslant 1} \frac{1}{1-z^m} = \sum_{n\geqslant 0} p(n) z^n$$

& extended partial fraction decompositions to higher order poles

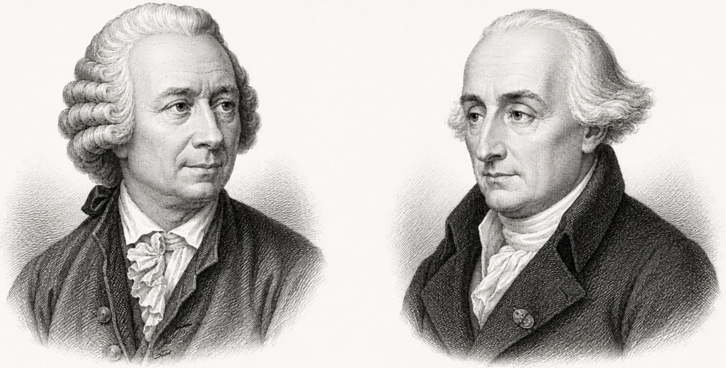
- **Lagrange (1736–1813), *algebraic series machinery*: symbolic use**

$$f(x+t) = (1+\Delta)^t f(x); \qquad f(x+t) = e^{t\partial_x} f(x)$$

& inversion of implicit series

$$y = z\phi(y) \quad \Rightarrow \quad [z^n]y^k = \frac{k}{n}[t^{n-k}]\phi(t)^n$$

Euler: object centered; Lagrange: operation driven



Leonhard Euler
1707–1783

Joseph-Louis Lagrange
1736–1813

GENERATING FUNCTIONS: LAPLACE'S SYNTHESIS

Laplacean pattern: ***Problem*** $\rightarrow \int \rightarrow$ ***Asymptotics***

From GF manipulations to asymptotics

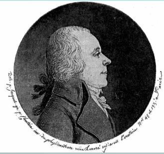
- **Laplace (*Théorie analytique des Probabilités*, 1812): *generating functions* (& the underlying *integral representations*) as a general, systematic calculus**
- **Probability $=$ algebra on generating functions**

$$X+Y:\ F_X(t)F_Y(t) \Longrightarrow X_1+\cdots+X_n:\ F_X(t)^n$$

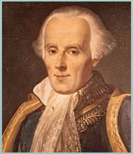
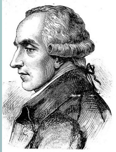
- **Difference/differential equations $\Rightarrow$ algebraic equations on GFs (or $\int$ transforms)**

La nature se rit des difficultés de l'intégration.

Pierre-Simon Laplace:
Pioneer of Asymptotic Methods





(1749–1827)

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- **1812–1820: *Théorie analytique des probabilités* (3 versions)**
- **1799: Minister of the Interior (6 weeks) by Napoléan**

  Napoléon criticized: "a mathematician of first rank, Laplace never proved himself a more than mediocre administrator. . . . Laplace never grasped issues from a proper perspective; he sought quibbles and saw problems everywhere, bringing the spirit of infinitesimals into the realm of governance"

  Roger Hahn's (2005) more reliable account: "the defeated emperor who dictated this assessment in Saint Helena was still smarting from what he felt was Laplace's disloyal conduct in 1814 when the Senate removed Naploléon as head of state, and in 1815 when Laplace failed to rally to his cause during the Hundred Days."

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BOOKS & RESOURCES

- I. Todhunter, *A history of the mathematical theory of probability: from the time of Pascal to that of Laplace*, 1865
- H. Andoyer, *L'œuvre scientifique de Laplace*, 1922
- S. M. Stigler, *The History of Statistics*, 1986
- C. C. Gillispie, *Pierre-Simon Laplace, 1749–1827*, 1997
- A. Hald, *A History of Mathematical Statistics from 1750 to 1930*, 1998
- A. I. Dale, *A History of Inverse Probability*, 1991; 2nd ed. 1999
- R. Hahn, *Pierre Simon Laplace, 1749–1827: a determined scientist*, 2005
- J. Dhombres & C. Alvarez, *Pierre Simon de Laplace, 1749–1827: Le parcours d'un savant*, 2012
- P. Gorroochurn, *Classic Topics on the History of Modern Mathematical Statistics: From Laplace to More Recent Times*, 2016

English translation webpages maintained by R. J. Pulskamp

Pierre Simon Laplace on Probability and Statistics

LAPLACE'S 1774 MÉMOIRE (25 yrs old)

The first use in history of the saddle-point method

Mémoire sur la prob. des causes par les événements

Asymptotics of Beta integral:

$$\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\int_{\frac{p}{p+q}(1-w)}^{\frac{p}{p+q}(1+w)} x^{p-1}(1-x)^{q-1}\,\mathrm{d}x \quad (\textbf{saddle-point method})$$

$$\sim 1-\frac{\sqrt{q}}{\sqrt{2\pi p(p+q)}\,w}\left((1+w)^p(1-w)^q+(1-w)^p(1+w)^q\right)$$

It is arguably the most influential article in this field to appear before 1800, being the first widely read presentation of inverse probability and its application to both binomial and location parameter estimation.

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REAL $\int$: LAPLACE'S METHOD (SADDLE-POINT)

$$\int_a^b h(x)e^{Nf(x)}\,dx \sim h(r)e^{Nf(r)}\sqrt{\frac{2\pi}{-Nf''(r)}} \qquad (f'(r)=0,\ f''(r)<0)$$

first use

1774
1st local–max estimate
inverse prob. of errors

Applications
probability / CLT
large powers asymp.
finite differences
observational sci.

book consolidation

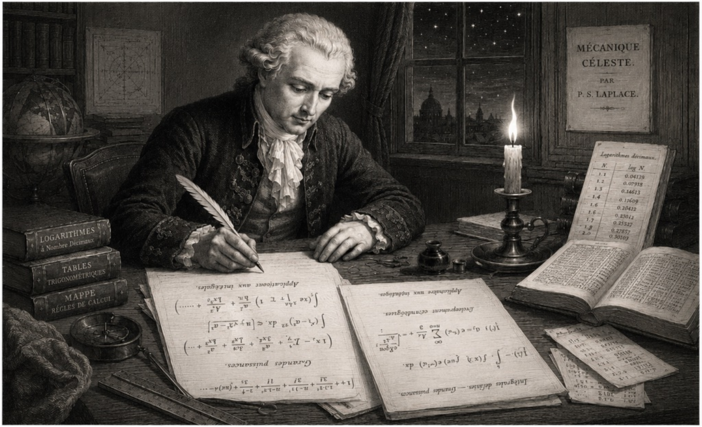
1812 / 1814 / 1820
Th. analytique des proba.
book-form reproduction,
supplements

1770 1780 1790 1800 1810 1820

1785–1786
general asymptotic method:
large powers, real integrals
applications

method + applications

QUILL PEN ERA: 6th – 1820s C



GFs & Laplace's method: worked out stroke by stroke with a feather.

LAPLACE'S 1785 & 1786 MÉMOIRES

Mémoire sur les approximations des formules qui sont fonctions de très grands nombres (130 pp.; 44 §§)

$\Rightarrow$ ***Th. & apps. of Laplace's method (for real $\int$)***

"très grands nombres" $\Rightarrow$ "high powers"

- **§1–7, general description of L's method**
- **§8–18, *solving finite difference equations by integral transforms (L. & Mellin)***
- **§19–31, applications:** $\Gamma(n)$ **(§19–§20),** $(s-m)y_{s+1} + sy_s = p^m$ **(§21),** $(s+1)y_{s+1} = 2(1+2s)y_s$ **(§22),** $[z^n](z^{-r} + \cdots + 1 + z + \cdots + z^r)^n$ **(§23), saddle-point eq. (§24), L & Mellin transf. (§25)**
- **§32–43, *statistical inference***
- **§26, §27, §44, high-order diff.** $\Delta^k 0^n$, $n \to \infty$ **&** $1 \leqslant k \leqslant n$

LAPLACE'S 1785 & 1786 MÉMOIRES

§26, §27, §44 & *TAP, 1820*: Livre I-3-§40 & Livre II-2-§4

$k!\left\{ {n \atop k} \right\}$	$=$	$\Delta^k 0^n$	$=$	$\sum_{0\leqslant j\leqslant k}\binom{k}{j}(-1)^j(k-j)^n$
ordered Stirling2		**finite diff.**		**balls in urns**

For large $n-k$ (saddle-point method)

$$(\star)\qquad \left\{ {n \atop k} \right\} \doteq \frac{n!R^{-n}(e^R-1)^k}{k!\sqrt{2\pi\beta_2 k}}\left(1-\frac{1}{k}\left(\frac{5\beta_3^2}{24\beta_2^3}-\frac{\beta_4}{8\beta_2^2}\right)+\cdots\right)$$

$$\rho := \tfrac{n+1}{k},\ \boxed{\rho = \frac{R}{1-e^{-R}}},\ \beta_j := j![y^j]\log\tfrac{e^{R(1+y)}-1}{(1+y)^{\rho}},\ \boxed{\beta_2 = \rho(R+1-\rho)}$$

For small $n-k$ (divergent Gamma integrals)

$$\left\{ {n \atop k} \right\} \doteq \binom{n}{k}\left(\frac{k}{2}\right)^{n-k}\left(1+\frac{1}{3k}\binom{n-k}{2}+\frac{5k-2}{15k^3}\binom{n-k}{4}+\cdots\right),$$

LAPLACE'S 1785 & 1786 MÉMOIRES

§26, §27, §44 & *TAP, 1820*: Livre I-3-§40 & Livre II-2-§4

$k!\left\{ {n \atop k} \right\}$	$=$	$\Delta^k 0^n$	$=$	$\sum_{0\leqslant j\leqslant k} \binom{k}{j}(-1)^j(k-j)^n$
ordered Stirling2		**finite diff.**		**balls in urns**

For large $n-k$ (saddle-point method)

$$(\star)\qquad \left\{ {n \atop k} \right\} \doteqdot \frac{n!R^{-n}(e^R-1)^k}{k!\sqrt{2\pi\beta_2 k}}\left(1-\frac{1}{k}\left(\frac{5\beta_3^2}{24\beta_2^3}-\frac{\beta_4}{8\beta_2^2}\right)+\cdots\right)$$

$$\rho := \frac{n+1}{k}, \boxed{\rho = \frac{R}{1-e^{-R}}}, \beta_j := j![y^j]\log\frac{e^{R(1+y)}-1}{(1+y)^\rho}, \boxed{\beta_2 = \rho(R+1-\rho)}$$

For small $n-k$ (divergent Gamma integrals)

$$\left\{ {n \atop k} \right\} \doteqdot \binom{n}{k}\left(\frac{k}{2}\right)^{n-k}\left(1+\frac{1}{3k}\binom{n-k}{2}+\frac{5k-2}{15k^3}\binom{n-k}{4}+\cdots\right),$$

LAPLACE'S 1785 & 1786 MÉMOIRES

§26, §27, §44 & *TAP, 1820*: Livre I-3-§40 & Livre II-2-§4

$k!\left\{{n \atop k}\right\}$	$=$	$\Delta^k 0^n$	$=$	$\sum_{0\leqslant j\leqslant k}\binom{k}{j}(-1)^j(k-j)^n$
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$\left\{ {n \atop k} \right\}$ BORN IN THE 18TH CENTURY

de Moivre (1712)
(Occupancy problem)

Stirling (1730)
(Finite differences)

$\left\{ {n \atop k} \right\}$ **in the 18th century**

Matsunaga, Arima, Saka
(Set partitions or Genjikō)

Laplace (1785–86)
(Asymptotic approximations)

Aspects: Combinatorial, Probabilistic, Numerical & Asymptotic

SADDLE-POINT METHOD IN 18TH CENTURY

Laplace's motivation: $\mathbf{P}$(all k tickets after n drawings)

- $a^{-n} = \dfrac{1}{\Gamma(n)} \displaystyle\int_0^\infty e^{-ax} x^{n-1}\, \mathrm{d}x \;(a, n > 0)$
- $\Delta^k a^{-n} = \displaystyle\sum_{0\leqslant j\leqslant k} \binom{k}{j} (-1)^j (k-j+a)^{-n} = \dfrac{1}{\Gamma(n)} \int_0^\infty x^{n-1} e^{-ax} (e^{-x} - 1)^k \, \mathrm{d}x$
- **formal saddle-point approximation:**

$$\Delta^k a^{-n} \sim \frac{e^{-aR+n}\left(e^{-R}-1\right)^k \left(\frac{n}{R}\right)^{1-n}}{\sqrt{n\left(\frac{n}{R^2} + \frac{ke^{-R}}{(e^{-R}-1)^2}\right)}} \quad \textbf{with} \quad \boxed{a = \frac{n-1}{R} + \frac{k}{e^R - 1}}$$

- **change** n **to** $-n$ **&** R **to** $-R$

$$\boxed{\Delta^k a^n \approx \frac{e^{aR-n}\left(e^{R}-1\right)^k \left(\frac{n}{R}\right)^{n+1}}{\sqrt{n\left(\frac{n}{R^2} - \frac{ke^{R}}{(e^{R}-1)^2}\right)}}} \quad \textbf{with} \quad \boxed{a = \frac{n+1}{R} - \frac{k}{1 - e^{-R}}}$$

perfectly correct & verified by numerical calculations

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What is interesting and characteristic of Laplace is that two false steps seem to cancel one with another. In ordinary mortals this might be ascribed to luck, but this sort of thing happens too often in Laplace to be ascribed to chance. — David & Barton (1962)

full justification only *176 years later*: Moser & Wyman (1958)

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APPLICATIONS (MOTIVATED BY LOTTERIES)

Laplace (1783): probability of full collection of k objects

take r each time: $$\sum_{0\leqslant j\leqslant k}\binom{k}{j}(-1)^{k-j}\frac{\binom{k-j}{r}^n}{\binom{k}{r}^n}$$

$$r=1: \quad p_n := \frac{k!}{k^n}\left\{ n \atop k \right\} = \frac{\Delta^k 0^n}{k^n} \approx e^{-kq}\left(1+\frac{k}{2n}q-\frac{n+k}{2}q^2\right)$$

$$(q := e^{-n/k})$$

Q: find the first n such that $p_n > \frac{1}{2}$ when $k = 10000$?

Laplace first got $n = 95768.5$, then corrected by 1 to $n = 95767.41$

This value may still differ by some units to the true one, but by correcting the value of $q = e^{-n/k}$ by the same procedure, . . . , which then gives, as the second value of n: $n = 95767.41$

Exact values: $p_{95767} \approx 0.49998\,58159\ldots$ **&** $p_{95768} \approx 0.50002\,04864\ldots$

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HOW DID LAPLACE ACHIEVE THAT IN THE 1780s?

$\log_{10} \left\{ {95767 \atop 10000} \right\} > 347408 \Rightarrow$ **no exact calculation possible**

How did he reach such a numerical precision in the 1780s?

- **First way: use a finer expansion**

$$S(n,k) := \left\{ {n \atop k} \right\} \frac{k!}{k^n} = e^{-\lambda} + \frac{\lambda e^{-\lambda}}{2k}(w-(w+1)\lambda) + \frac{\lambda e^{-\lambda}}{24k^2} \begin{pmatrix} -w(3w-8)+21w^2\lambda \\ -2(9w^2+11w+4)\lambda^2 \\ +3(w+1)^3\lambda^3 \end{pmatrix} + \cdots$$

(n,k)	Dominant term	2 terms	3 terms
$(95767, 10^4)$	0.49994 68716	0.49998 57653	0.49998 58158
$(95768, 10^4)$	0.49998 15301	0.50002 04359	0.50002 04864

- **Second (modern) way: use the saddle-point expansion** ($\star$)

$$S(95767, 10^4) = 0.49998\,62797(1 + 4.29307\,70452 \times 10^{-8})$$

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Laplace (1812, French lottery): what is the smallest n s.t.

$$Q: \sum_{0 \leqslant j \leqslant k} \binom{k}{j} (-1)^j \prod_{0 \leqslant i \leqslant 4} \left(1 - \frac{j}{k-i}\right)^n > \frac{1}{2} \text{ when } k = 90?$$

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LAPLACE'S EXPANSION WHEN $k \sim n$

Method of divergent Gamma integrals

1. $$k!\left\{ {n \atop k} \right\} = \Delta^k 0^n = \frac{\int t^{-n-1}(e^{-t}-1)^k \,\mathbf{d}t}{\int t^{-n-1} e^{-t} \,\mathbf{d}t}$$
2. $$(e^{-t}-1)^k = (-t)^k e^{-kt/2}\left(1 + \frac{kt^2}{24} + \frac{k(5k-2)t^4}{5760} + \cdots\right)$$
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"two false steps seem to cancel one with another"

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"two false steps seem to cancel one with another"

. . . it is remarkable how much is "discovered" today which can be found, if not in de Moivre's Doctrine of Chances, in Laplace's Théorie Analytique des Probabilités.

— David & Barton (1962)

LAPLACE QUOTES

- *Read Euler, read Euler. He is the master of us all.*
- **What we know here is very little, but what we are ignorant of is immense.**
- *The most important questions of life are indeed, for the most part, really only problems of probability.*
- **Probability theory is nothing but common sense reduced to calculation.**
- *The language of analysis, the most refined of all, is itself a powerful tool for discovery; its notations, when necessary and well designed, often give rise to new branches of calculus.*

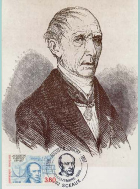
19TH C: SINGULAR $\int \Rightarrow$ COMPLEX ANALYSIS

Cauchy's justifications:

$\Rightarrow$ ***From real to complex***

$\Rightarrow$ ***Foundation of complex analysis***

Augustin-Louis Cauchy (1789–1857)



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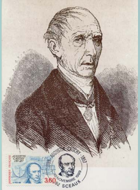

Augustin-Louis Cauchy (1789–1857)

AUGUSTIN-LOUIS CAUCHY

- **1789: born in Paris (becoming extreme royalist & Catholic)**
- **1815–1816: professor (at ÉP) & elected member (*Acad. des Sci.*)**
- **1821 & 1823: "Analyse algébrique" & "Calcul infinitésimal" (rigorous foundation of calculus; *algebra of ε-δ inequality*)**
- **1825: his most important work on *foundation of complex analysis***
- **1830–1838 (July Revolution): voluntary exile to Turin & Prague**
- **1838–1848: back to Paris (no position but the most productive)**
- **1849–1857: professor (U Paris)**
- **1857: died in Sceaux**

Only periods with academic position: 1815–1830 & 1849–1857

Œuvres complètes: 27 volumes ($>$ 13000 pages)

Belhoste (1991): *...such an enormous scientific creativity is nothing less than staggering, for it presents research on all the then-known areas of mathematics: arithmetic, algebra, geometry, statistics, mechanics, real and complex analysis, and mathematical physics.*

Cauchy: An anti-revolutional revolutionist

Cauchy's life & French revolutions

Years	Cauchy's Life	French history
1789-1792	Born in Paris	Monarchy ⇒ Revolution
1792-1804	Fleeing Paris & back	⇒ 1st Republic
1804-1814	Lycée ⇒ Engineer	⇒ 1st Empire
1814-1830	Professor (1815)	Napoléan → Bourbon
1830-1848	Self-exiled & no position	July Revolution Louis-Philippe: King
1848-1857	Professor	February Revolution 2nd Republic
1857	Dies in Sceaux	Napoléon III's 2nd Empire

A life deeply intertwined with political and religious currents

Useful books

- Grabiner (1981), *The origins of Cauchy's rigorous calculus*
- Belhoste (1991), *Augustin-Louis Cauchy: a Biography*
- J. Mawhin (1995), *Augustin-Louis Cauchy: l'itinéraire tourmenté d'un mathématicien légitimiste [the tumultuous journey of a legitimist mathematician]*
- Smithies (1997), *Cauchy and the creation of complex function theory*
- Bottazzini and Gray (2013), *Hidden harmony—geometric fantasies, the rise of complex function theory*
- Gray (2015), *The real and the complex: a history of analysis in the 19th century*
- Bradley & Sandifer (2009), *Cauchy's Cours d'analyse an annotated translation*
- Cates (2019), *Cauchy's Calcul Infinitésimal, an annotated English translation*

Freudenthal (2008) stated: More concepts and theorems have been named for Cauchy than for any other mathematician.

- Smithies (1997): *In 1826 Cauchy began to publish his Exercices de Mathématiques, which was essentially a mathematical periodical consisting entirely of papers written by himself; it appeared at approximately monthly intervals until 1830.*

- The *Nouveaux Exercices de Mathématiques* were published in Prague in 1835–36 and the *Exercices d'Analyse et de Physique Mathématique* appeared at intervals from 1840 to 1853.

- Belhoste (1991): *. . . publishing a periodical that was devoted solely to his own research studies, an unheard-of privilege in the history of mathematics.*

- Grabiner (1981), 18th-c calculus (use, heuristic) vs 19th-c calculus (rigorous justification): *Cauchy separated the task of discovering results by means of . . . from real to complex ones—from the quite different task of proving theorems.*

N. H. Abel (1826): *Cauchy is mad, and there is no way of being on good terms with him, although at present he is the only man who knows how mathematics should be treated. What he does is excellent, but very confused . . .*

Cauchy's 1815 mémoire (published in 1844): justifying Laplace's analysis for $\Delta^n a^k$

Mémoire sur diverses formules relatives à la théorie des intégrales définies et sur la conversion des différences finies des puissances en intégrales de cette espèce, [submitted to the Académie on January 2, 1815], Journal de l'École polytechnique, 17 (1844), 147–248.

From $\Delta^k a^n$ to singular integrals

- **From finite differences to integrals**

$$\Delta^k a^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1} e^{-sx} (e^{-x} - 1)^k \, \mathbf{d}x$$

- **From $-n$ to n (analytic continuation)**

$$\Delta^k a^n = \frac{1}{\Gamma(n)} \int_0^\infty x^{-n-1} e^{-sx} (e^{-x} - 1)^k \, \mathbf{d}x?$$

Q: *how to deal with this singular integral when $n \geqslant 0$?*

Analytic continuation of Gamma-integral

Laplace used integration by parts: for $-m < \Re s < 1 - m$

$$\Gamma(s) = \int_0^\infty \left(e^{-x} - \sum_{0\leqslant j<m} \frac{(-x)^j}{j!}\right) x^{s-1}\,\mathbf{d}x$$

implying that $a^s = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-ax} - \sum_{0\leqslant j<m} \frac{(-ax)^j}{j!}\right) x^{-s-1}\,\mathbf{d}x$

Then from real x ***to complex*** $z = x + iy$

$$\Delta^k 0^n = \frac{n!}{\pi} \int_0^\infty \frac{(e^{2c} - 2e^c \cos x + 1)^{k/2}}{(c^2 + x^2)^{(n+1)/2}} \cos\big((n+1)t - k\theta\big)\,\mathbf{d}x$$

where $t = \arctan \frac{x}{c}$ **and** $\theta = \arctan \frac{\sin x}{\cos x - e^{-c}}$**.**

Then apply Laplace's saddle-point method.

1815 ($\mathbb{R} \to \mathbb{C}$) $\to$ 1825 (Complex analysis) $\to$ 1844 (published)

Q: why he didn't use $\Delta^k a^n = \frac{n!}{2\pi i} \oint_{|z|=c} z^{-n-1} e^{az} (e^z - 1)^k\,\mathbf{d}z$

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$$\Gamma(s) = \int_0^\infty \Big(e^{-x} - \sum_{0\leqslant j<m} \frac{(-x)^j}{j!}\Big) x^{s-1}\,\mathbf{d}x$$

implying that $a^s = \dfrac{1}{\Gamma(-s)} \displaystyle\int_0^\infty \Big(e^{-ax} - \sum_{0\leqslant j<m} \frac{(-ax)^j}{j!}\Big) x^{-s-1}\,\mathbf{d}x$

Then from real x ***to complex*** $z = x + iy$

$$\Delta^k 0^n = \frac{n!}{\pi} \int_0^\infty \frac{(e^{2c} - 2e^c \cos x + 1)^{k/2}}{(c^2+x^2)^{(n+1)/2}} \cos\big((n+1)t - k\theta\big)\,\mathbf{d}x$$

where $t = \arctan \dfrac{x}{c}$ **and** $\theta = \arctan \dfrac{\sin x}{\cos x - e^{-c}}$**.**

Then apply Laplace's saddle-point method.

1815 ($\mathbb{R} \to \mathbb{C}$) $\to$ 1825 (Complex analysis) $\to$ 1844 (published)

Q: why he didn't use $\Delta^k a^n = \dfrac{n!}{2\pi i} \displaystyle\oint_{|z|=c} z^{-n-1} e^{az} (e^z - 1)^k\,\mathbf{d}z$

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Cauchy (1829): Mémoire sur divers points d'analyse

$$\int_0^\infty g(x)f(x)^n\,\mathbf{d}x, \quad [z^m]g(z)f(z)^n, \quad [z^n]g(z)f(z)^n$$

The last two pages contain a post-scriptum on the asymptotics of $\Delta^k s^n$.

Cauchy (1828): Sur les différences finies des puissances entières d'une seule variable

Residue algebra: for $k = n - \ell$, ℓ small

$$\Delta^k 0^n = n![z^n](e^z - 1)^k = \binom{n}{k}\left(\frac{k}{2}\right)^{n-k}\left(1 + \frac{(n-k)(n-k-1)}{6k} + \cdots\right)$$

Complete Laplace's analysis for $\Delta^k 0^n$

Gray (2015): In the great Mémoire (1825) he produced the integral and residue theorems that are today named after him, but even then he does not seem to have appreciated the depth of his discovery, perhaps because he was still not thinking *geometrically* about complex variables . . . by the *1870s* complex function theory was on its way to becoming one of the dominant fields of mathematics.

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SADDLE-POINT METHOD IN THE 20TH CENTURY

use of SPM becoming prevalent

- ***Analytic (Cauchy integral or Laplace integral)*:**
  - **Bleick & Wang:** $R^{-n} \int (1+iy)^{-n-1} (e^{R(1+iy)} - 1)^k \, \mathrm{d}y$
  - **Arfwedson, Moser & Wyman:** $r^{-n} \int e^{-int} (e^{re^{it}} - 1)^k \, \mathrm{d}t$
  - **Change of variables:** $\int u^{k-1} \log(1+u)^{-n} \, \mathrm{d}u$
  - **Bender:** $[y^k z^n] \frac{1}{1+y(e^z-1)}$
  - **Two-dimensional Cauchy + saddle**
  - **Temms's uniform asymptotic approximation**
  - **. . .**
- ***Probabilistic (sum of independent RVs)*:**
  - **d'Ocagne, Harper: all roots of** $\sum_k \left\{ {n \atop k} \right\} x^k$ **real** $\Rightarrow$ **sum of Bernoulli RVs**
  - **Conditional probabilities**
  - **Urn models**
  - **. . .**

SADDLE-POINT METHOD IN THE 21ST CENTURY

H. (2023): $\boxed{n+1 \text{ vs } n}$**? or** $\boxed{\int_{c-i\infty}^{c+i\infty} \text{ vs } \oint_{|z|=r}}$

Laplace $\int$: $\rho_1 := \frac{n+1}{k}$, $R = \rho_1(1-e^{-R})$, $V_1(R) = \rho_1 k(r+1-\rho_1)$

$$\boxed{\left\{ {n \atop k} \right\} \sim \frac{n!}{k!} \cdot \frac{R^{-n}(e^R-1)^k}{\sqrt{2\pi V_1(R)}} \sum_{m\geqslant 0} \frac{(-1)^m(2m)!}{m!2^m} d_m(r) V_1(R)^{-m}}$$

$$\boxed{d_m(R) = [y^{2m}]\left(\frac{\frac{1}{2}\rho_1(R+1-\rho_1)y^2}{\log(1+\frac{\rho_1}{R}(e^{Ry}-1)) - \rho_1\log(1+y)}\right)^{m+\frac{1}{2}}}$$

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$$\boxed{\left\{ {n \atop k} \right\} \sim \frac{n!}{k!} \cdot \frac{r^{-n}(e^r-1)^k}{\sqrt{2\pi V(r)}} \sum_{m\geqslant 0} \frac{(-1)^m(2m)!}{m!2^m} c_m(r) V(r)^{-m}}$$

$$\boxed{c_m(r) = [t^{2m}]\left(\frac{\frac{1}{2}\rho(r+1-\rho)t^2}{\log(e^{re^t}-1) - \log(e^r-1) - \rho t}\right)^{m+\frac{1}{2}}}$$

Cauchy quotes

- ***Men pass away, but their deeds abide.***
- ***I get up every morning at four o'clock and I am busy from that time on . . . I never get tired of working, on the contrary, it revives me and I am in perfect health . . .***

Laplace is real

Laplacean pattern

$$\textit{Problem} \to \int \to \textit{Asymptotics}$$

Cauchy style rigorization

$$\textit{Problem} \to \oint \to \textit{Asymptotics}$$

Cauchy is complex

If Laplace needed a philosophical creed, it could have been

“from phenomena to laws, and from laws to causes".

— Roger Hahn (2005)

For Cauchy, we could use:

“from rigor to absolutes, and from absolutes to faith".

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For Cauchy, we could use:

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徐志摩: 數大 便是美

(Beauty grows with magnitude)